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Pattern Selection: Determined by Symmetry and Modifiable by Distant Effects

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We consider Saffman–Taylor channel flow without surface tension on a high-pressure driven interface, but modify the usual infinite-fluid in infinite-channel configuration. Here we include the treatment of efflux by considering a finite connected body of fluid in an arbitrarily long channel, with its second free interface the efflux of this configuration. We show that there is a uniquely determined translating solution for the driven interface, which is exactly the $1/2$ width S–T solution, following from correct symmetry for a finite channel flow. We establish that there exist no perturbations about this solution corresponding to a finger propagating with any other width: Selection is locally unique and isolated. The stability of this solution is anomalous, in that all freely impressible perturbations are stabilities, while unstable modes request power proportional to their strength from the external agencies that drive the flow, and so, in principle, are experimentally controllable. This is very different from the behavior of the usual infinite fluid. We conjecture that surface tension on the *efflux* interface modifies channel-width λ according to $1-2\lambda=\sigma/v$ (i.e., $(2\pi)^2 B$ of the literature) with v the velocity of the high-pressure tip, but σ the surface tension of the efflux. That is, λ is decreased below $1/2$ by the effect of smoothing the distant efflux. The perturbation theory created here to deal with transport between two free boundaries is novel and dependent upon a symmetry implied by the equations of motion.

KEY WORDS: Pattern selection; boundary conditions at infinity; Hele-Shaw flow; Saffman–Taylor fingers.

1. INTRODUCTION

Objects arising from flows, aggregation and so forth often have distinctive shapes. (Think, for example, of snowflakes.) Upon theoretical study, (the

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identification of all forces and processes at work, and the determination of all boundary conditions), the solution will be that of this particular shape. In other cases, however, the usual form is just one of many possible solutions that can be found. The question of what then distinguishes and determines only the usual, observed form is termed a problem of "pattern selection." Clearly something has been left out of the theoretical treatment, which means that after a careful treatment, something apt to be subtle remains to be dealt with. "Left out" itself is subtle, since the theoretical considerations, and their solutions, entail approximations, and the "truth" can have been left out by the approximations. We consider such a problem here.

Perhaps the earliest studied selection problem arose in the 1958 work of Saffman–Taylor (S–T).[2] Here the question is the shape of a bubble penetrating into a body of viscous fluid. Loosely, a soda-straw is filled with a thick fluid which is then forcibly blown out. How has the breath of air propagated down the straw, to drive out what amount of the fluid? Performed carefully, the straw of length l is flattened down all along its length into a long rectangle of width w and length l, where $2w$ is just about the circumference of the cylindrical straw, and a channel of thickness $b \ll w$ all that remains of the third dimension of the straw's volume. The material of the "straw" is transparent glass for visualization. That is, there are two w by l plates of glass, one a gap b above the other, and the long edges are sealed. View the w by l rectangles from above with the long length horizontal (the x-axis) and the width w along the y-axis. A controlled source of high pressure air is now applied to the left end (say, at $x = 0$), and the driven viscous fluid flows out the right end into atmosphere (the efflux). The bubble of air advancing from the left into the viscous fluid, the immiscible high-pressure interface between the driving air and viscous fluid, hereafter referred to as H, is the object of study. (See Fig. 1, but imagine the viscous fluid to continue beyond L.)

Invariably, S–T observed the bubble to develop into a very long "finger," that is, a rounded nose furthest to the right with long horizontal sides parallel to the long sides of the channel trailing back towards the high pressure end. This finger is almost perfectly centered within the channel, that is, with center-line symmetry. Moreover, S–T discovered that with

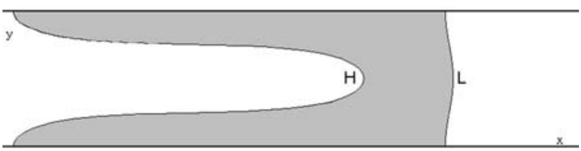

Fig. 1. The channel in (x, y) coordinates. The body of viscous fluid is shaded. It is bounded between a high-pressure interface H with "air" and another, L at low-pressure.

higher and higher pressures, and hence increasingly faster bubbles, the width of the horizontal sides of the finger approached 1/2 the width w of the channel. This is the selected pattern for this problem.

S–T then theoretically determined H of finger shape, but of width λw with λ anywhere from 0 to 1, rather than just 1/2. They had no argument as for why only the observed value should occur, and so exposed a problem of pattern selection.

Let us be more careful and say more about the experiment. There is no reason to use just air to drive the viscous fluid. Indeed, any fluid immiscible with the fluid initially filling the channel may be employed to drive it. The problem is simplest if the driving fluid, like air, has a negligible viscosity, so that pressure is spatially uniform throughout it, and hence all along the interface H. However, there is always some surface tension along H, so that when H is curved, *just* within the viscous fluid along H, pressure is *not* uniform. It is straightforward to see (say on dimensional grounds) that this surface tension σ can modify the problem only in the combination σ/v with v a velocity, say the velocity of the tip of the finger. It then follows that the selection of 1/2, occurring for high velocities, is selection in the limit $\sigma/v \to 0^+$. This then sets up the question of how this *singular* limit accounts for the observed selection. It is singular because, as determined in the 1958 paper, with $\sigma \equiv 0$, there is no selection at all.

There is an extensive literature, culminating in the mid-1980s and employing "beyond all orders" of perturbation methodology to verify this singular limit, and hence the critical role played by even the slightest of surface tensions acting along the boundary of the selected pattern.[4–6] It is now the accepted wisdom that all these indeterminate selection problems achieve their theoretical resolution through additional "constituentive" relationships (such as σ for S–T) imposed directly upon the shape to be so selected.

Let us again be more careful, and say more about the theory. The gap b is viewed as so small, that with a usual limit of the viscous fluid equations, this "film" of fluid can be treated purely within 2-D. This is a theoretically compelling circumstance, since the problem then rapidly is noticed to be fully amenable to complex, conformal mathematics.[7, 14] Apart from some studies as to how finite values of b can modify the 2-D solution, the entire literature on S–T flow is 2-D conformal.

The theoretical and experimental literature nevertheless diverge in an obvious way. While discussing surface tension on the bubble interface, the experimental literature is relatively vague about how the viscous fluid emerges from the apparatus at low pressure. It is not that this is totally unimportant—care has to be paid to terminations, and the reliable reproducibility of fast long fingers emerges only after some adjustment of the efflux termination. Perhaps it is largely undocumented in consequence of an implicit belief that far enough away from the observed interface, these details should significantly have decoupled.

The theoretical literature has gone much further. The *entire* literature considers only $l = \infty$, namely, an infinitely long channel all filled with the viscous fluid. This is evidently a very different geometry from that of the experiments that determine selection. Granted that subtleties can potentially lead to selection or its absence, one might wonder that this approximation to very large length has an impact upon selection. This question has never been asked in the literature. We do so here, and indeed discover that the limit $l \to \infty$ is also singular, in that significant phenomena exist for any finite length but not in the infinite geometry.

Since conformal mathematics is so powerful, we consider a finite termination of geometrically simplest form. Namely, we allow the channel containing the fluid to be arbitrarily long, while the viscous fluid, always within it, is of a fixed finite volume. Abutting the fluid at its low pressure, right end, we again have air; that is, a fluid with negligible viscosity, as is the high pressure driving fluid. Hence, we now have two interfaces; the high-pressure, experimentally observed one H, and the low-pressure, efflux one at the right, which we now term L. Both interfaces are dynamically free surfaces, although they are evidently highly coupled, since between them resides the connected body of our incompressible viscous fluid. (See Fig. 1.)

In this finite configuration, the pressure drop across the viscous fluid becomes physically finite, whereas it is simply infinite in the usual treatment. There is also a net flux of fluid. In the infinite treatment it freely can be set at any value with impunity, and is conventionally set at the constant 1. In our finite case, the impedance of the flow, the ratio of net pressure drop by flux is again finite and large, as opposed to simply infinite in the usual treatment. More importantly, this impedance generally changes in time, since the closest distance between the two interfaces dynamically changes in time, and generally decreases as the interface H grows more bent into a finger. In experiment ref. 3, the time derivative of impedance is indeed significantly negative. Although impedance variations are buried in an infinite value in the usual infinite treatment, it is troublesome that there is no trace of its derivative in that literature.

The reason we have considered a finite body of fluid with definite pressures along its free boundaries is that there is a mathematical clue to anticipate an influence upon shape. The theoretical works on the infinite fluid all consider periodic boundary conditions for the cross-channel (y) behavior. The channel has two long sides a distance w apart, one edge at $y = 0$ and another at $y = w$. Interface H (the only one for the infinite problem) has viscous fluid everywhere to its right in the direction of increasing x (and hence decreasing pressure, p, with $p \to -\infty$ as $x \to +\infty$). The literature takes p periodic in y with period w so the x-values of H at $y=0$ and $y=w$ agree. The edges are impenetrable with no separation of fluid from them, so that the velocity of fluid along them is purely in the x-direction, and so, $y=0$ and $y=w$ are streamlines for the flow. Periodicity of w, in synchronizing the flow at both edges is clearly a higher symmetry than required. It is readily seen that this implies center-line symmetry, with v also in the x-direction along $y=w/2$. This, however, is just the symmetry of the observed finger, and so the symmetry in which to seek a theoretical result. This then is the choice of the literature.

Implicit in this choice of symmetry is, of course, the underlying fact that the flow is along the x-direction at each wall. This has striking implications. The conformal machinery utilizes an invertible analytic mapping $\zeta = h(z)$ from the fluid's spatial plane $z \equiv x+iy$ to a new plane $\zeta \equiv -p+is$ with p pressure, and s the "stream function," where the curves of constant s are always orthogonal to those of constant pressure. The thin limit of the viscous fluid equations is Darcy's law: The velocity is the negative gradient of pressure. Thus, fluid flows with a velocity at a point, at each moment of time, in the direction of a curve of constant s through that point at that time. (These curves all evolve in time as then does h.) In particular, with velocity in the x-direction along the walls, this is to say real in the complex description, since Darcy's law reads $v_x - iv_y = h'(z)$. This implies that at each instant of time the part of the strip between $y=0$ and $y=w$ all to the right of H is mapped by h into a strip between $s=0$ and $s=w$ all to the right of $p=0$ by the obvious choice of the origin of pressure (only differences matter). But then, the real z-axis is mapped by h to the real ζ-axis. This, in consequence of analytic continuation, implies that h is Schwarz reflection symmetric, $\bar{h}(\bar{z}) = h(z)$. (This, together with an identical statement for the upper edge implies periodicity over $2w$, the true symmetry of the flow. Increasing the symmetry to w periodicity is equivalent to an extra reflection symmetry through $y=w/2$.)

The equation of motion for the interface H is that it flow into itself under its velocity at each instant as determined by Darcy's law. Expressed in the ζ-plane, this becomes a relationship along H (which is just $\mathrm{Re}\,\zeta = 0$) of both the ζ and time derivatives of $f = h^{-1}$ and their complex conjugates. With f reflection symmetric, these conjugates are then just the functions themselves evaluated at the conjugate of ζ. But, along H, the conjugate of ζ is simply $-\zeta$. Thus, along H the equation of motion is an analytic partial differential equation in ζ and time. But then it analytically continues as that same equation as a field equation throughout fluid and its continuation.[1, 13] That is, there is a new symmetry of the equations of motion, *parity* in ζ. The symmetry of $s \to -s$ is the periodicity of the literature.

Unnoticed in the literature, but implicitly present, is the symmetry of $p \to -p$. That is, the equations of motion express a relationship between positive and negative values of p. This is extraordinary, since it expresses some connection between the shape of H, determined by singularities in f at (positive) high pressures to behaviors of the fluid at (negative) low pressures, i.e., to its efflux. This paper explores that connection, and discovers it be replete with consequences.

In particular, for a translating finger, $\lambda = 1/2$ is uniquely determined for the finite geometry with pressure fixed on each interface just within the viscous fluid, and so with zero surface tension on both H and L. This immediately implies that the limit $l \to \infty$ is singular, and S–T pattern selection is unambiguously resolved in finite geometry.

The class of S–T solutions for arbitrary λ in the infinite problem is particularly simple. When we consider how they transport fluid particles arbitrarily far downstream, it is easily discovered that only the $\lambda = 1/2$ solution has cross-channel curves of particles all at a fixed value of pressure all flowing into other curves of fixed pressure at later times. With no surface tension, the fluid may be opened up to atmosphere along any such curve, which then is L. The actual form of the $\lambda = 1/2$ solution came from mathematical simplicity in the infinite problem. In the finite version it is the *unique* solution to the question of a translating finger. The other S–T solutions for $\lambda < 1/2$ far downstream (far to the right) correspond to solutions for the finite problem with H still without surface tension, but with $\sigma/v = 1 - 2\lambda$ on L, the curve that can then be opened to atmosphere. This confirms the significant connection between the shape of H and some behavior (smoothing) of L. Moreover only $\lambda < 1/2$ is physically allowed, but requiring enormous values of σ to substantially decrease λ from $1/2$. For viscosities and high velocities of the experimental literature, this amounts to no more than a several percent reduction in λ. Reference 3 found just such an order of magnitude reduction from $1/2$, which there was interpreted as the absence of any genuine selection, and speculated that it might be in consequence of films of fluid left behind on the glass plates behind the advancing finger. Moreover, it violates the singular limit theory of $\sigma \to 0^+$, which precisely determines just $1/2$. Finiteness can possibly also contribute to this experimental observation.

Next, with $\sigma = 0$ on H for the infinite literature, all the S–T solutions are linearly unstable. The $\sigma \to 0^+$ theory stabilizes its $\lambda = 1/2$ solution. The finite problem without surface tension, significantly, has its $\lambda = 1/2$ solution stable in a somewhat anomalous sense. When we consider fluctuations about a given solution, we presume that they can be imposed with an arbitrarily small expenditure of power. In the finite problem the same instabilities, that in the $\sigma = 0$ infinite geometry grew exponentially, require power exactly proportional to their size in order to be sustained. If there is no source to provide this power, then the solution is fully stable in finite geometry. This means that if the pump driving the flow, say at fixed pressure, is prevented, by some feedback control, from engaging in rapid variations in the power requested from it, then the solution is stabilized. This is so different from the infinite geometry, that we again realize $l \to \infty$ is singular. The $l \to \infty$ limit of the $\sigma \to 0^+$ theory has never been worked out, let alone questioned. It clearly should be.

The idea that the treatment of an arbitrarily distant efflux can modify the shape of an object has never been experimentally checked. It is worth bearing in mind that aggregation arises in diffusive processes for which again there is an incompressible fluid with identical dynamics—namely the probability flux of random walkers. There is enough sufficiently delicate in the considerations to follow in this paper, that one can only go so far as to speculate that quite to the contrary of the accepted wisdom, patterns might more generally depend on details remote from the selected shape itself.

This paper is laid out as follows: In Section 2 we erect the conformal field equations suitable to two free interfaces, and notice that they are displacement invariant under a time φ, the integrated flux up to time t. In order to determine exact solutions to perturb about, we ask the natural questions: Are there purely translating solutions, and are there solutions which automatically satisfy the equations of *both* interfaces, simply in consequence of this translation invariance?

In Section 3 we establish that the only such solutions turn out to be the $1/2$ width Saffman–Taylor fingers, here not guessed at, but uniquely determined. We determine that Saffman–Taylor fingers of a narrower width correspond to gathering and smoothing the efflux in a precise way, and no matter how distant that efflux is from the driven interface.

Section 4 develops the machinery to explicitly solve the finite problem in perturbation theory. A felicitous symmetry of the equations of both interfaces allows us to fully integrate both equations into a purely algebraic problem. Although it is known that the infinite case is fully integrable, that is not known in the finite geometry we discuss. This symmetry allows one to speculate that this problem too is completely integrable. A result of this effort is that all unstable modes, and only these, determine (unstably growing) external impedances, and this one function, just of time, uniquely encodes the entire unstable spatial flow, thus allowing a pump driving the flow to "hear" the precise shape of the interface simply by detecting the power requested from it, and so, rendering it capable of completely stabilizing the flow. No such thing is true in the infinite case, and so we realize that just how the flow is terminated matters, no matter how long the body of fluid.

In Section 5 we realize that while formal perturbative pole-dynamical solutions[9–12] that apparently produce different width fingers exist, these solutions are *purely* formal, and can't correspond to any actual approximate solutions in the finite problem *nearby* to the $1/2$ finger. This fully establishes a locally isolated pattern selection that can be stable under an appropriate control. This control, however, is likely to become exquisitely delicate as the length of the fluid diverges.

The Discussion Section 6 more iconically presents what this paper has accomplished after the reader has been armed by the machinery presented.

The paper ends in an Appendix that establishes that the pure S–T $1/2$ finger is unique, and so totally isolated within pole-dynamics for the finitely terminated problem. It has been relegated to an appendix in order not to interrupt the main line of exposition. This section contains some modest new results for pole-dynamics, and is a self-contained extension to paper.[1] That paper can also assist the reader in gaining familiarity with the workings of our reflection symmetric field equations. Indeed, some of the material of this section was reflected into that paper.

2. THE GENERAL SETUP AND EQUATIONS OF MOTION

Taking $z=x+iy$, with x increasing down-stream along a channel of cross-width $y\in[0,\pi]$, the fluid obeys Darcy's law for the velocity v and pressure p

$$v=-\partial p. \tag{2.1}$$

The flow is incompressible so that $\partial\cdot v=0=\Delta p$, and so p is harmonic, and the real part of an analytic function h, naturally constructed through its derivative:

$$\begin{aligned} h' &\equiv \frac{\bar v}{V(t)}=\frac{v_x-iv_y}{V(t)}=\frac{-\partial_x p+i\partial_y p}{V(t)} \\ h &\equiv \xi+i\eta=\frac{-p+is}{V(t)}=\zeta, \end{aligned} \tag{2.2}$$

which satisfies Cauchy–Riemann for $\Delta p=0$, and s is the harmonic conjugate of $-p$. $V(t)$ here is a function of time alone. Consider the change in h along an arbitrary curve γ

$$\delta h=\int_\gamma h'\,dz=-\frac{1}{V}\,\delta p+\frac{i}{V}\int_\gamma v_n\,d\ell. \tag{2.3}$$

Since by Eq. (2.2) $\delta\xi = -\delta p/V(t)$,

$$\delta\eta = \frac{\pi}{V(t)}\left(\frac{1}{\pi}\int_\gamma v_n\, d\ell\right)$$

and so

$$\delta\eta \equiv \pi \qquad \text{for} \quad V(t) = \frac{1}{\pi}\int_\gamma v_n\, d\ell. \tag{2.4}$$

Thus with $V(t)$ the mean velocity of all fluid (the conserved flux divided by cross-width π), the original spatial channel is analytically mapped to an identical one in ζ. (Although flux is conserved throughout the incompressible fluid, for a finite system it will generally vary in time.)

Assuming the fluid not to stagnate other than in front of added boundaries, $v \neq 0$, $h' \neq 0$, and so h is invertible, *conformally* mapping one $[0, \pi]$ strip onto another. We denote its inverse as f:

$$f = h^{-1}, \quad f' = \frac{V}{\bar{v}} \qquad \text{at} \quad z = f(\zeta). \tag{2.5}$$

The virtue of f is that all boundaries are known in ζ: The problem we consider has impenetrable walls at $\eta = 0, \pi$ (along which v is real) and two free interfaces, one at $p \equiv 0$, i.e., $\xi = 0$ and the other at $p = p_{\text{atm}} \equiv -p_g$, or at

$$\xi = \xi_g \equiv p_g/V. \tag{2.6}$$

$p_g > 0$ is the gauge pressure across the body of fluid, and ξ_g (the total impedance) positive and generally time-dependent, because either or both of p_g and V are. (p_g is time dependent if we experimentally modulate the driving pressure.) Thus, in ζ, the fluid is just a rectangle extending from $\xi = 0$ up to a variable right hand edge.

To track a fluid particle through use of the one-parameter (time) family of maps, f, tag a particle at $t = 0$ by its ζ-coordinate ζ_0:

$$z(\zeta_0, t) = f(\zeta(\zeta_0, t), t); \qquad \zeta(\zeta_0, 0) \equiv \zeta_0. \tag{2.7}$$

Then

$$v = z_t = f'\zeta_t + f_t = V/\bar{f}', \qquad \text{or} \quad V = |f'|^2\, \zeta_t + f_t \bar{f}'. \tag{2.8}$$

Once f and V are known, Eq. (2.8) is an ordinary differential equation to be solved for $\zeta(\zeta_0, t)$ given the initial data of Eq. (2.7).

To determine f requires the physical input that an interface simply transports under its velocity. On an interface $\operatorname{Re}\zeta = \hat{\xi}(t)$, with $\hat{\xi} = 0$ on the left interface and $\hat{\xi} = \xi_g(t)$ on the right. That is, if $\operatorname{Re}\zeta_0 = \hat{\xi}(0)$ then $\operatorname{Re}\zeta(\zeta_0, t) = \hat{\xi}(t)$ for a fluid particle on the interface. That is,

$$\operatorname{Re}\zeta_t = \dot{\hat{\xi}}(t) \qquad \text{on} \quad \operatorname{Re}\zeta = \hat{\xi}(t), \tag{2.9}$$

so that (2.8) is

$$V(t) = \operatorname{Re}(\bar{f}'(f'\dot{\hat{\xi}} + f_t)) \qquad \text{on} \quad \operatorname{Re}\zeta = \hat{\xi}, \tag{2.10}$$

on each interface. The form of (2.10) recommends the following definition:

$$f_{\hat{\xi}}(\zeta, t) \equiv f(\zeta + \hat{\xi}(t), t), \tag{2.11}$$

immediately yielding

$$V(t) = \operatorname{Re}(\bar{f}'_{\hat{\xi}} f_{\hat{\xi}, t}) \qquad \text{on} \quad \operatorname{Re}\zeta = 0 \tag{2.10$'$}$$

on each interface $\operatorname{Re}\zeta = \hat{\xi}(t)$.

The generally unknown-in-advance $V(t)$ complicates (2.10′). Delaying its determination to a final step, it is overwhelmingly expedient to now eliminate it through a redefinition of time:

$$\dot{\varphi} \equiv V, \tag{2.12}$$

so that φ-time is integrated flux to time t, that is, the volume of fluid moved, or the energy expended by external sources in the case $p_g \equiv \text{const}$. But then $\partial_t = V\partial_\varphi$ and (2.10′) is simply

$$1 = \operatorname{Re}(\bar{f}'_{\hat{\xi}} f_{\hat{\xi}, \varphi}) \qquad \text{on} \quad \operatorname{Re}\zeta = 0. \tag{2.10$''$}$$

Finally since f takes $\operatorname{Im}\zeta = 0$ to $\operatorname{Im} z = 0$, f is Schwarz-reflection symmetric,

$$f(\bar{\zeta}, t) = \overline{f(\zeta, t)}. \tag{2.13}$$

But $\operatorname{Re}\zeta = 0 \to \bar{\zeta} = -\zeta$, and (2.10″) reads

$$2 = f'_{\hat{\xi}}(-\zeta)\, f_{\hat{\xi}, \varphi}(\zeta) + f'_{\hat{\xi}}(\zeta)\, f_{\hat{\xi}, \varphi}(-\zeta). \tag{2.14}$$

Although (2.14) is exactly (2.10″) on $\operatorname{Re}\zeta = 0$, (2.14) is in fact correct over all of f's analytic continuation, since (2.14) is an equation just of the analytic variable ζ and (2.14) and (2.10″) agree all along the curve of an interface.

For our problem of two free interfaces, let us call $f_{\xi=0}$ simply f, and $f_{\xi_g} \equiv g$:

$$g(\zeta, \varphi) = f(\zeta + \xi_g(\varphi), \varphi). \tag{2.15}$$

Then the equations of motion (2.14) are simply

$$2 = f'(-\zeta)\, f_\varphi(\zeta) + f'(\zeta)\, f_\varphi(-\zeta), \tag{2.14$'$}$$

$$2 = g'(-\zeta)\, g_\varphi(\zeta) + g'(\zeta)\, g_\varphi(-\zeta). \tag{2.14$''$}$$

That is, f and its ξ_g translate g are *both* solutions to (2.14′). Although (2.14′) is strange in its ζ-dependence, it is simply autonomous (translation-invariant) in φ. That is,

$$f(\zeta, \varphi) \quad \text{a solution to (2.14}') \Rightarrow f(\zeta, \varphi + \varphi_0) \quad \text{also a solution.} \tag{2.16}$$

This is significant in that purely translating solutions are allowed and a path is also opened for *both* f and g to obey (2.14′) in a way independent of the matching-up of inner details, merely reliant on translation invariance.

Pause to notice that reflection symmetry for f has determined a new symmetry, parity, $\zeta \to -\zeta$, for the equations of motion (2.14′). This entails not only a cross-channel symmetry, but also a relation of p to $-p$. But then, far upstream singularities which determine the shape of the driven interface become related to far downstream properties, which is to say how the flow is terminated. This observation is the foundation for the work that follows.

Not only is f reflection symmetric about the ξ-axis, but also about $\operatorname{Im} \zeta = \pi$ since the upper horizontal wall there is also impenetrable. Coupled with f's reflection symmetry, this is simply the periodicity of f':

$$f(\zeta + 2\pi i) = f(\zeta) + 2\pi i.$$

Reflection symmetry itself means that both the channel and its ζ-image may be mentally reflected through the x and ξ-axes respectively, so that, in light of the above, both views are periodic in $y \in [-\pi, \pi]$, although only the upper half is physical. However, the experimental flow actually *is* very close to symmetric about its center line. We now capitalize on this by taking the physical channel to be $[-\pi, \pi]$, so that reflection symmetry now enforces actual symmetry, and later, if we so choose, study the stability of this symmetry.

That is, our channel is now of width 2π and Eq. (2.4) is for the sequel modified to

$$V(t) = \frac{1}{2\pi} \int_{\gamma} v_n \, d\ell.$$

3. PATTERN SELECTION AND PHYSICAL FINITENESS

Since the equations for the interfaces are φ-translation invariant, within an available infinite channel we can seek solutions for which the driven ($p = 0$) interface retains its shape, but merely translates in φ-time. Such an f must be of form

$$f = \beta(\varphi) + u(\zeta). \tag{3.1}$$

(For clarity of exposition, we suspend the demonstration of this until after (3.19).) Then, by (2.14′)

$$\frac{1}{\beta'(\varphi)} = \frac{u'(\zeta) + u'(-\zeta)}{2} \equiv \lambda \tag{3.2}$$

for λ some real constant. Neglecting any branching considerations and taking u analytic at $\zeta = 0$, (3.2) integrates to

$$\frac{u(\zeta) - u(-\zeta)}{2} = \lambda\zeta \tag{3.3}$$

for the antisymmetric part of u. On $\xi = 0$, with u reflection symmetric

$$\operatorname{Im} u(i\eta) = y(\eta) = \lambda\eta$$

so that the driven interface occupies a fraction λ of the channel width. Thus $\lambda \leqslant 1$. With $\lambda < 1$ the interface must stretch off to minus infinity. By “finite” we shall then request that at least the right interface (the “efflux”) be of bounded x-extent. These solutions with $\lambda < 1$ are called fingers in consequence of their long asymptotically horizontal sides at $\pm i\pi\lambda$. (We will determine truly finite configurations at the end of this section.)

By (3.2), (3.3)

$$f = \frac{\varphi}{\lambda} + u(\zeta) \tag{3.4}$$

with

$$u(\zeta) = \lambda\zeta + E(\zeta); \qquad E(-\zeta) = E(\zeta). \tag{3.5}$$

Clearly E must have a discontinuity of $2\pi i(1-\lambda)$ throughout fluid to meet channel wall data, so that E must contain logarithms. Otherwise neither λ nor E is determined.

Next

$$g = \frac{\varphi}{\lambda} + u(\zeta + \hat{\xi}(\varphi)) \tag{3.6}$$

must again satisfy (2.14′):

$$2 = (u'_+ + u'_-)/\lambda + 2\hat{\xi}' u'_+ u'_- \qquad \text{with} \quad u'_\pm = u'(\pm\zeta + \hat{\xi}).$$

Multiplying by $2\lambda^2/u'_+ u'_-$ and rearranging,

$$F(\hat{\xi}+\zeta)\, F(\hat{\xi}-\zeta) = 1 + 4\lambda^2\hat{\xi}'(\varphi) \tag{3.7}$$

with

$$F(\zeta) \equiv 2\lambda/u'(\zeta) - 1. \tag{3.8}$$

Substituting (3.8) into the second of (3.2),

$$F(\zeta)\, F(-\zeta) = 1 \tag{3.9}$$

so that

$$F(0) = \pm 1. \tag{3.10}$$

Setting $\zeta = \hat{\xi}$ in (3.7), at which it must be defined,

$$F(\hat{\xi}+\zeta)\, F(\hat{\xi}-\zeta) = \pm F(2\hat{\xi}) \tag{3.11}$$

and

$$1 + 4\lambda^2\hat{\xi}' = \pm F(2\hat{\xi}). \tag{3.12}$$

There are two possibilities: $\hat{\xi}$ varies with φ or is just a constant. If $\hat{\xi}' = 0$ then (3.7) and (3.9) become

$$F(\zeta + 2\hat{\xi}) = F(\zeta)$$

and hence u' is doubly periodic with real period $2\hat{\xi}$ and imaginary period $2\pi i$. But then, by (3.5) $E'(\zeta)$ is an odd such doubly periodic function. Not to be just constant (and hence zero), E' must have singularities in each cell. They cannot lie in $\xi \in (0, \hat{\xi})$ where f must be analytic. But then, they can't lie in $(\hat{\xi}, 2\hat{\xi})$ either, since then, by periodicity they lie in $(-\hat{\xi}, 0)$ and then by oddness, in $(0, \hat{\xi})$ again. Thus singularities must appear on either $\xi = 0$ or $\xi = \hat{\xi}$. With $\lambda < 1$, there must be simple poles to pick up the requisite $2\pi i(1-\lambda)$ discontinuity. With no net integral around a cell, the sums of the residues must vanish, and so there must be at least two poles per cell. Finally, the cut in E must run the length of fluid, and so there must be poles on both $\xi = 0$ and $\xi = \hat{\xi}$. But then, the right interface must also be unbounded, which we have rejected by finiteness. That is, we have

$$\lambda = 1, \qquad f = \varphi + \zeta \tag{3.13}$$

or $\hat{\xi}' \neq 0$.

If $\hat{\xi}' \neq 0$, the only continuous solution to (3.11) with $\hat{\xi}$ and ζ independently varying is exponential:

$$F(\zeta) = \pm e^{-n\zeta} \tag{3.14}$$

with n an integer to meet $2\pi i$ periodicity. But then,

$$\begin{aligned} u' &= \frac{2\lambda}{1 \pm e^{-n\zeta}}, \\ u &= \frac{2\lambda}{n} \ln(e^{n\zeta} \pm 1) = 2\lambda\left(\zeta + \frac{1}{n}\ln(1 \pm e^{-n\zeta})\right). \end{aligned} \tag{3.15}$$

Thus, $n \geqslant 1$ (fluid to the right of ξ) *and* $\lambda = 1/2$, so that

$$f = 2\varphi + \zeta + \frac{1}{n}\ln(1 \pm e^{-n\zeta}) \tag{3.16}$$

and wall boundary geometry is satisfied, since the coefficient of ζ is 1. $n > 1$ is simply n parallel fingers each of width $1/2n$, and for simplicity, and the one finger case we care about from experiment,

$$f = 2\varphi + \zeta + \ln(1 + e^{-\zeta}) \tag{3.17}$$

where we have chosen the $+$ to have the nose of the finger at channel center. This is *precisely* the $1/2$ width Saffman–Taylor finger. However, both

$\lambda = 1/2$ and the precise form of E are here totally and uniquely determined. That is, with the efflux physically treated (the right interface at atmospheric pressure) pattern selection is complete. (The great bulk of this paper consists in comprehending its stability.)

Returning to (3.12) to determine $\hat{\xi}$,

$$\hat{\xi}' = -1 + e^{-2n\hat{\xi}}$$

or,

$$e^{2n\hat{\xi}} = 1 + ke^{-2n\varphi}$$

or, with $n = 1$,

$$e^{2\hat{\xi}} = 1 + ke^{-2\varphi} \tag{3.18}$$

where k determines, at reference time $\varphi = 0$, the length L_0 of fluid from the center of the driven interface at ln 2 to the center of the right interface at $\ln(e^{\hat{\xi}(0)} + 1)$, or

$$k = 4e^{2L_0}(1 - e^{-L_0}). \tag{3.19}$$

Let us finish this deduction by showing that f must indeed be of form (3.1). Call the fixed translating curve of H $z = z_*(l)$, uniquely parametrized by arclength l. Generally, u should be of the form $u(\zeta, \varphi)$, so that on H, $\zeta = i\eta$ and $u(i\eta, \varphi) = z_*(l(\eta, \varphi))$. Should l's parametrization on η *not* be φ dependent, then this $u(i\eta)$ is independent of φ as is then its unique analytic continuation, producing form (3.1). Differentiating on η and φ and substituting these in the equation of motion for the interface, $\mathrm{Re}(\bar{f}' f_\varphi) = 1$ on $\zeta = i\eta$ integrates to $\eta = \beta'(\varphi)\, y_*(l(\eta, \varphi)) + \alpha(\varphi)$. By reflection symmetry, y_* changes sign with η, and so the integration constant α identically vanishes. Next, as η varies over 2π, for all φ, for the fixed-shape interface, y_* varies by $2\pi\lambda$ with λ its constant channel width. This immediately determines $\beta' = 1/\lambda$ and $\lambda\eta = y_*(l(\eta, \varphi))$. But now, with z_* possessing a horizontal tangent at just isolated points, or asymptotically as l diverges, l must be independent of φ, and so f is of the form (3.1).

Before further considering same details of this solution, and rendering the solution truly finite, let us consider what we have accomplished, and why efflux treatment determines selection. We proceed in two steps. First, we consider a class of "solutions" containing (3.17) in order to see just how sharply $\lambda = 1/2$ is selected, and how it is contingent upon the treatment of efflux. In the bulk of this paper that follows, we consider if (3.1), exact translation, isn't simply too precise and perhaps not quite physical a request. (We shall vindicate it.)

By (3.17) and (3.5),

$$E = \ln 2 \cosh \zeta/2,$$

and so (2.14′) for just the driven interface is satisfied by

$$f = \varphi/\lambda + \lambda\zeta + 2(1-\lambda)\, E$$

or,

$$f = \varphi/\lambda + \zeta + 2(1-\lambda)\, \ln(1+e^{-\zeta}), \tag{3.20}$$

for any $\lambda \in (0, 1)$. These are the Saffman–Taylor fingers for width λ. As we already know, there is no ξ_g for any $\lambda \neq 1/2$ for which g will also satisfy (2.14′). Let us see why.

To track fluid particles under (3.20), we determine $\zeta(\zeta_0, \varphi)$ by (2.8):

$$(\xi' + i\eta')\, |1 + (2\lambda - 1)\, e^{-\zeta}|^2 = \left(\frac{1}{\lambda} - 1\right)(e^{-2\xi} - 1 + 2ie^{-\xi} \sin \eta). \tag{3.21}$$

It is immediately clear that the equation decouples in ξ if and only if $\lambda = 1/2$, in which case

$$\xi' = e^{-2\xi} - 1$$

(just (3.18), of course) and fluid particles all with the same value of ξ at time φ, but with all different η values, flow again at later times into all with the same value of ξ. This is so no matter how far downstream we look. And just so, for $\lambda \neq 1/2$, no matter how far downstream, a curve of fluid particles all at the same ξ at time φ will fail to be so at any later time. That is, by opening the channel to the atmosphere all along the curve $\xi = \hat{\xi}(\varphi)$ for $\lambda = 1/2$, the flow will continue to remain at atmospheric pressure. However for $\lambda \neq 1/2$, this flow may *never*, no matter how far downstream, be able to be opened to atmosphere.

But with f differing from free flow by $O(e^{-\xi})$, perhaps our question of $\lambda = 1/2$ is purely mathematical, and not physical. Let us now see that it is decidedly physical and independent of ξ, with measurable consequences, indeed consequences that conceivably have already been experimentally measured, and in the literature.

By (3.21)

$$\frac{d\xi}{d\eta} = -\frac{\sinh \xi}{\sin \eta}$$

and so,

$$\tanh\frac{\xi}{2}\tan\frac{\eta}{2}=\tan\frac{\theta}{2} \tag{3.22}$$

for that curve $\zeta(\varphi)$ which at $\xi\to+\infty$ ends at $\eta=\theta$. (3.22) has the series development

$$\eta=\theta+\sum_{1}\frac{2}{n}e^{-n\xi}\sin n\theta. \tag{3.23}$$

We proceed now to solve (3.21) just far downstream, retaining terms just of $O(e^{-\xi})$:

$$\xi'(1+2(2\lambda-1)\,e^{-\xi}\cos\eta)\sim-\left(\frac{1}{\lambda}-1\right). \tag{3.24}$$

By (3.23) to $O(1)$,

$$\cos\eta\sim\cos\theta$$

and integrating,

$$\xi-2(2\lambda-1)\,e^{-\xi}\cos\theta\sim\xi_0(\theta)-\left(\frac{1}{\lambda}-1\right)\varphi\equiv\hat{\xi}(\varphi,\theta)\qquad \xi_0\gg1$$

and so,

$$\xi\sim\hat{\xi}(\varphi,\theta)+2(2\lambda-1)\,e^{-\hat{\xi}}\cos\theta, \tag{3.25}$$

and by (3.23),

$$\eta\sim\theta+2e^{-\hat{\xi}}\sin\theta. \tag{3.26}$$

Next, to $O(e^{-\xi})$,

$$z=f\sim\zeta+2(2\lambda-1)\,e^{-\zeta}+\varphi/\lambda$$

and so, substituting (3.25), (3.26),

$$z\sim\xi_0+\varphi+i\theta+2\lambda e^{-\hat{\xi}+i\theta}, \tag{3.27}$$

and at each time φ, θ parameterizes a curve of fluid particles spanning the channel. For ξ_0 large enough, at $\varphi=0$ these are just a flat interface by selecting those fluid particles with $\xi_0(\theta)\equiv\xi_0$ independent of θ.

Let us now compute the curvature of $z(i\theta)$. This is just, with the center of the curvature to the left,

$$\kappa = \frac{1}{|z'|} \operatorname{Re}\left(\frac{z''}{z'}\right).$$

With $z' = 1 + 2\lambda e^{-\hat{\xi}+i\theta}$, to $O(e^{-\xi})$,

$$\kappa = 2\lambda e^{-\hat{\xi}} \cos\theta \tag{3.28}$$

and so, (3.25) reads

$$\hat{\xi}(\varphi) \sim \xi + \left(\frac{1}{\lambda} - 2\right)\kappa. \tag{3.29}$$

Consider now *treating* the efflux, namely endowing the right efflux interface with a surface tension σ, and maintaining the pressure against this interface constant along it. Then,

$$\hat{p}(\varphi) = p - \sigma\kappa$$

and dividing by $-V(\varphi)$,

$$\hat{\xi}(\varphi) = \xi + \frac{\sigma}{V}\kappa.$$

But, this is precisely (3.29) with

$$\frac{1}{\lambda} - 2 = \frac{\sigma}{V}, \tag{3.30}$$

and this result no matter how asymptotically far downstream we open the flow to atmosphere. Now, f' of (3.20) at $\zeta = 0$ (the nose of the finger) is λ, and so, the mean V is related to the finger's speed v by $V = \lambda v$. Then (3.30) is

$$1 - 2\lambda = \frac{\sigma}{v} \equiv (2\pi)^2 B \sim 40B \tag{3.31}$$

with the dimensionless B that of the literature

$$B^{-1} = 12\,\frac{\mu v}{\sigma}\left(\frac{w}{b}\right)^2$$

with μ viscosity, w/b the ratio of channel width to the gap between the upper and lower plates, and 2π comes from our natural choice of channel width, rather than 1.

We now see that with $\sigma > 0$, we can only produce fingers of width smaller than $1/2$ by physical smoothing and "gathering" means, and that the surface tension must be physically enormous to significantly decrease λ from $1/2$. For the physical choices of σ at experimental conditions when one observes σ/v so small as to be close to zero surface tension, one has $B \sim 10^{-3}$. Had this been our configuration, with no surface tension on the driven interface, but this value for the efflux interface, we would produce a $\lambda \sim 0.48$. Because we now realize that how the *efflux* is treated does matter, and since we have no idea just how it was accomplished in the published literature (because the experimentalists presumed it didn't matter), all we can say is that we have explained both qualitatively and to within order of magnitude a puzzle in the literature.[3]

Just as importantly, we have determined that the infinite channel-full of fluid is physically an incomplete specification of an experiment. In never physically treating efflux, the pole at $\operatorname{Re}\zeta \to +\infty$ of (3.20) promiscuously allows the physical completion of an experiment in any way whatsoever. For a given physical method of collecting efflux, the fluid can be opened to air at any large, but nevertheless, finite length down the channel. This limit can then be taken to infinity, but for example, in our configuration with a second interface, leaving intact the selection of (3.31). That is, the pole at infinity is actually, and physically illegally, within fluid, and so the usual theory of the infinite channel is too compliant to diverse stresses.

Since finiteness matters, it is useful to determine a finite equivalent to (3.1). We can utilize φ-invariance more deeply by requesting that the second interface is none other than the first, but delayed in time by φ_0:

$$f(\zeta + \hat{\xi}(\varphi, \varphi_0), \varphi) = f(\zeta, \varphi - \varphi_0) + \beta(\varphi_0). \tag{3.32}$$

By this we mean first that a constant, $\beta(\varphi_0)$ is allowed since (2.14′) entails just derivatives. Clearly, the amount of delay is related to the length of the body of fluid, so that the impedance $\hat{\xi}$ at the right interface depends upon it as well. We seek solutions for any chosen length of fluid, and so for arbitrary φ_0. Finally, we seek a shape of the driven interface that, in the spirit of (3.1) is independent of just what length we contemplate so that f itself is independent of φ_0. As the length of the body of fluid goes to zero, as with $\hat{\xi}$ of (3.18) with $k \to 0$, we request

$$\hat{\xi}(\varphi, 0) \equiv 0 \Rightarrow \beta(0) = 0. \tag{3.33}$$

Let us now determine the solutions to (3.32).

Set $\varphi_0 = \varphi$ in (3.32) and define

$$\hat{\xi}(\varphi, \varphi) \equiv \zeta_*(\varphi); \qquad f(\zeta, 0) \equiv u(\zeta)$$

then,

$$f(\zeta + \zeta_*(\varphi), \varphi) = \beta(\varphi) + u(\zeta)$$

or,

$$f(\zeta, \varphi) = \beta(\varphi) + u(\zeta - \zeta_*(\varphi)). \tag{3.34}$$

Substituting in (3.32), with

$$\gamma(\varphi, \varphi_0) \equiv \hat{\xi}(\varphi, \varphi_0) - \zeta_*(\varphi) + \zeta_*(\varphi - \varphi_0),$$

obtain

$$u(\zeta + \gamma(\varphi, \varphi_0)) - u(\zeta) = \beta(\varphi - \varphi_0) - \beta(\varphi) + \beta(\varphi_0).$$

Differentiating on ζ, should γ vary with φ, φ_0 then $u' \equiv \text{const}$, and $u = \zeta$. Otherwise, with γ constant, setting $\varphi_0 = 0$, by (3.33) $\gamma \equiv 0$, in which case $\beta(\varphi - \varphi_0) = \beta(\varphi) - \beta(\varphi_0)$ and so $\beta(\varphi) = \varphi/\lambda$ for some constant λ. Thus,

$$f(\zeta, \varphi) = \frac{\varphi}{\lambda} + u(\zeta - \zeta_*(\varphi)) \tag{3.35}$$

and

$$\hat{\xi}(\varphi, \varphi_0) = \zeta_*(\varphi) - \zeta_*(\varphi - \varphi_0). \tag{3.36}$$

We now ask that (3.35) satisfies (2.14′). But this is precisely the discussion for g of (3.6) upon replacing $\hat{\xi}(\varphi)$ by $-\zeta_*(\varphi)$, but without (3.9) so that $F(0) \equiv a$ is unknown. Certainly $\zeta_*' \neq 0$, since otherwise $\hat{\xi} \equiv 0$ by (3.36). So, by (3.15),

$$u(\zeta - \zeta_*) = 2\lambda\left(\zeta - \zeta_* + \frac{1}{n}\ln(1 + ae^{-n(\zeta - \zeta_*)})\right)$$

with $ae^{-n(\zeta-\zeta_*)} \equiv \pm e^{-n(\zeta-\hat{\zeta})}$ and $\hat{\zeta} < 0$, we must again have $\lambda = 1/2$ for channel geometry. (3.12) is now

$$\zeta_*' = 1 - a^2 e^{2n\zeta_*}$$

or,

$$\hat{\zeta}' = 1 - e^{2n\hat{\zeta}}$$

or,

$$e^{-2n\hat{\zeta}} = 1 + e^{-2n\varphi}$$

together with

$$f = 2\varphi + \frac{1}{n}\ln(e^{n(\zeta-\hat{\zeta})} \pm 1). \tag{3.37}$$

But this is just the simplest pole-dynamical $1/2$ finger, converging to (3.16) as $\varphi \to +\infty$. Indeed, (3.16) is the limit of these finite solutions as the length $\varphi_0 \to +\infty$ *and* we take $\varphi \to +\infty$, but leaving a finite length of fluid downstream.

Thus, with $n = 1$ and the $+$ sign, and in the above limit of a truly finite problem, we have precisely two solutions related to translation:

$$\lambda = 1, \qquad f = \varphi + \zeta, \qquad \hat{\xi} = L_0 \equiv \text{const}, \qquad e^{2\hat{\xi}} \equiv k \tag{3.13}$$

and

$$\lambda = 1/2, \qquad f = 2\varphi + \ln(e^{\zeta} + 1), \qquad e^{2\hat{\xi}} = 1 + ke^{-2\varphi}. \tag{3.17}$$

For a fully finite version, we have

$$\begin{gathered} \lambda = 1/2, \qquad f = 2\varphi + \ln(e^{\zeta-\hat{\zeta}} + 1), \qquad e^{-2\hat{\zeta}} = 1 + e^{-2\varphi}, \\ \hat{\xi} = \hat{\zeta}(\varphi) - \hat{\zeta}(\varphi - \varphi_0), \qquad e^{2\hat{\xi}} = \frac{1 + ke^{-2\varphi}}{1 + e^{-2\varphi}}, \end{gathered} \tag{3.38}$$

with

$$\varphi_0 = L = A/2\pi, \qquad A = \text{area of fluid}, \qquad k = e^{2\varphi_0}.$$

4. REFLECTION SYMMETRIC PERTURBATION THEORY FOR TWO INTERFACES

Having asked, in the light of experiment, for purely translating solutions, we must now answer the question if the fluid in this theory of no surface tension actually assumes the selected solution, or instead, even when initially set, deeply unstably runs away from them. There is a related question to this. For $L_0 \gg 1$ in (3.19) and $\varphi \ll L_0$, by (3.18)

$$\hat{\xi} \sim L_0 - \varphi + \ln 2$$

and $\hat{\xi}$ together with the length of fluid downstream from the driven interface are very large. By the time there has been a net flow of fluid $\sim L_0$, $\hat{\xi} \sim 1$ and the right interface has begun to bend into a $1/2$ finger parallel to the left interface, with the distance between the two interfaces rapidly turning into a thin film for larger values of φ. It is intuitive that a long body of fluid downstream can help stabilize the driven interface, but certainly not just a film of fluid. That is, one can expect qualitatively different behaviors with $\varphi \ll L_0$ and $\varphi \simeq L_0$. Let us call this former regime the "finger regime." This raises a new question. Is it possible that during the finger regime we have solutions exponentially close (i.e., $\sim e^{-L_0}$) to purely translating, but then rapidly, for $\varphi \sim L_0$ becoming highly dynamical. Then, from the physical viewpoint of an experiment, we observe a "perfectly" translating finger for the duration of observation. This makes (3.1) again too sharp a question, and the theory might well allow for f's similar to those of (3.20) for arbitrary λ, but decorated by dynamical corrections of $O(e^{-L_0})$ over the finger regime.

In a way, such a possibility is curious. It implies that for (3.20) a powerful surface tension is requested on the efflux, while (3.20) decorated by $O(e^{-L_0})$ corrections produces exponentially close motion throughout the great bulk of the fluid, while requiring no surface tension at all. This is surprising since by (3.25), for any λ,

$$\hat{\xi} \sim L_0 - \left(\frac{1}{\lambda} - 1\right)\varphi,$$

so that macroscopically different fluxes are elicited without physical agency.

Beyond our $\lambda = 1/2$ exact solutions, we know of no other simple, exact solutions with two interfaces. For example, it is not difficult to show that the pole dynamics of the literature is exhausted by just (3.38). (Since the details of this exercize are tangential to our main thrust here, it has been relegated to an appendix, Appendix A.) This leaves us in the position for a perturbative treatment, which we now erect. (However, this does suffice for the determination of all solutions nearby to the purely translating one.)

Perturbing a solution to (2.14′) determines a very special change of variables which renders the calculations feasible. Donate by $\hat{f}$ a known solution to (2.14′), call its perturbed value f_p, the strength of perturbation ϵ, and f the perturbation:

$$f_p = \hat{f} + \epsilon f + \cdots. \tag{4.1}$$

We shall denote by "transpose" the symmetry operation of (2.14′):

$$f^t(\zeta, \varphi) \equiv f(-\zeta, \varphi), \tag{4.2}$$

so that (2.14′) for $\hat{f}$ is

$$\hat{f}'^t \hat{f}_\varphi + (\hat{f}'^t \hat{f}_\varphi)^t = 2. \tag{4.3}$$

Entering (4.1) into (2.14′) leads to $O(\epsilon)$:

$$\hat{f}'^t f_\varphi + \hat{f}^t_\varphi f' + \text{transpose} = 0,$$

or,

$$\hat{f}'^t f_\varphi + \hat{f}^t_\varphi f' = \tilde{a}(\zeta, \varphi); \qquad \tilde{a}^t = -\tilde{a}. \tag{4.4}$$

Integrating (4.4) on curves in a solution surface, with s a parameter,

$$\frac{d\varphi}{ds} = \hat{f}'(-\zeta, \varphi), \qquad \frac{d\zeta}{ds} = \hat{f}_\varphi(-\zeta, \varphi), \qquad \frac{df}{ds} = \tilde{a}(\zeta, \varphi). \tag{4.5}$$

By the first two,

$$0 = \hat{f}_\varphi(-\zeta, \varphi)\frac{d\varphi}{ds} - \hat{f}'(-\zeta, \varphi)\frac{d\zeta}{ds} = \frac{d}{ds}\hat{f}(-\zeta, \varphi). \tag{4.6}$$

That is, we should take a function purely of $\hat{f}^t$ as one new variable, and take a function of (ζ, φ) independent of $\hat{f}^t$ as s. Then, the partial derivative of f with respect to this second new variable is, by the third o.d.e., antisymmetric in ζ. This optimally selects the transpose of the first variable as the second variable. To ward off exponentials when we consider an $\hat{f}$ such as (3.17), we are led to define the following two new variables in place of (ζ, φ):

$$v \equiv e^{\hat{f}(-\zeta, \varphi)}, \qquad \xi \equiv e^{\hat{f}(\zeta, \varphi)}. \tag{4.7}$$

We then have by (4.2) for any function u,

$$u^t(v, \xi) = u(\xi, v), \tag{4.8}$$

the reason we have denoted the symmetry of (2.14′) as transpose. The last of (4.5) with $s \equiv \ln \xi$ is

$$\xi f_2(v, \xi) = \tilde{a}(v, \xi) \equiv \xi v a_{12}(v, \xi), \qquad a^t = -a$$

since the mixed derivative of an antisymmetric function is also antisymmetric, and so

$$f = v \partial_v a(v, \xi). \tag{4.9}$$

(The integration constant, a function of v is just an $a(v, \xi) = u(v) - u(\xi)$.) That is, the solution to first order perturbation, (4.4) are $\ln v$ derivatives of an arbitrary antisymmetric "potential" a, and first order theory for one interface has been fully integrated, *despite* the non-local (in ζ) form of the p.d.e. (2.14′).

Let us check that our new variables v, ξ are sensibly defined. Now, ζ is complex while φ is real. Both v and ξ are complex. If they are formally independent, then upon inverting they determine a complex extension of φ. But they are independent, precisely in virtue of (2.14′):

$$\frac{\partial(\ln \xi, \ln v)}{\partial(\zeta, \varphi)} = \begin{pmatrix} \hat{f}' & \hat{f}_\varphi \\ -\hat{f}'^t & \hat{f}^t_\varphi \end{pmatrix} \tag{4.10}$$

and so, by (2.14′)

$$\left| \frac{\partial(\ln \xi, \ln v)}{\partial(\zeta, \varphi)} \right| = 2. \tag{4.11}$$

Inverting (4.10),

$$\frac{\partial(\zeta, \varphi)}{\partial(\ln \xi, \ln v)} = \frac{1}{2} \begin{pmatrix} \hat{f}^t_\varphi & -\hat{f}_\varphi \\ \hat{f}'^t & \hat{f}' \end{pmatrix}. \tag{4.12}$$

Paying attention to (3.18), rather than φ, $e^{-2\varphi}$ will constantly appear, and so we choose for later a new time variable

$$\lambda \equiv e^{-2\varphi}, \tag{4.13}$$

for which we have by the second row of (4.12),

$$v \partial_v \lambda = -\lambda \hat{f}' \qquad \text{and} \qquad \xi \partial_\xi \lambda = -\lambda \hat{f}'^t \tag{4.14}$$

which will shortly become important.

Before proceeding it is useful to rewrite (2.14′) in (ν, ξ) variables. By (4.10),

$$\partial_\zeta = \hat{f}'\xi\partial_\xi - \hat{f}''\nu\partial_\nu, \qquad \partial_\varphi = \hat{f}_\varphi\xi\partial_\xi + \hat{f}^t_\varphi\nu\partial_\nu. \tag{4.15}$$

Substituting in (2.14′) for $\hat{f}$, then yields

$$\frac{1}{\nu\xi} = f_\xi f^t_\xi - f_\nu f^t_\nu. \tag{4.16}$$

Clearly $\hat{f} = \ln \xi$ obeys (4.16), while $i \ln \nu$ is not reflection symmetric. It is now trivial to obtain the perturbation results, which to second order are

$$f_p = \ln \xi + \epsilon f + \epsilon^2 F + \cdots \tag{4.17}$$

and so,

$$\frac{1}{\nu} f_\xi + \frac{1}{\xi} f^t_\xi = 0 \tag{4.18}$$

and

$$\frac{1}{\nu} F_\xi + \frac{1}{\xi} F^t_\xi = f_\nu f^t_\nu - f_\xi f^t_\xi. \tag{4.19}$$

(4.18) is, of course, the observations below (4.6), and its general solution just (4.9). It is also easy, by successive integrations by parts and the use of (4.18) to integrate (4.19) fully into

$$F = f\xi\partial_\xi f + \nu\partial_\nu(\tfrac{1}{2} f f^t + A) \tag{4.20}$$

with A another antisymmetric potential.

So far we have just discussed one interface. This suffices for the infinite channel. With the flat solution, (3.13),

$$\xi = e^{\zeta+\varphi}, \qquad \nu = e^{-\zeta+\varphi}$$

and so

$$a = \frac{\nu^p}{p} - \frac{\xi^p}{p} \Rightarrow f = \nu^p = e^{-p\zeta + p\varphi}$$

with p the positive integers is a full basis of perturbations for the one interface at Re $\zeta = 0$, and all are exponentially unstable.

For the 1/2 finger solution of (3.17) (and almost identically for (3.38)), with λ of (4.13),

$$\xi = \frac{1+e^{\zeta}}{\lambda}, \qquad v = \frac{1+e^{-\zeta}}{\lambda}$$

and again

$$a = \frac{v^p}{p} - \frac{\xi^p}{p} \Rightarrow f = v^p = (1+e^{-\zeta})^p\, e^{2p\varphi}$$

all with $p>0$ unstable. Here we can also take for the v term of $a \int \frac{dv}{v} \ln v$ producing

$$f = \ln v = 2\varphi + \ln(1+e^{-\zeta})$$

and so perturb to the nearby Saffman–Taylor solutions (3.20) with $\lambda = (1-\epsilon)/2$. Indeed, any function $u(v)$ analytic at Re $\zeta \to +\infty$ provides a legal perturbation.

Let us now go on to the second interface. Not only does $\hat{f}$ satisfy (2.14′), but so too does

$$\hat{g}(\zeta, \varphi) = \hat{f}(\zeta + \hat{\xi}(\varphi), \varphi). \tag{4.21}$$

Accordingly, we define

$$\tilde{v} \equiv e^{\hat{g}(-\zeta, \varphi)}, \qquad \tilde{\xi} \equiv e^{\hat{g}(\zeta, \varphi)} \tag{4.22}$$

so that, by (4.21),

$$\tilde{\xi}(\zeta, \varphi) = \xi(\zeta + \hat{\xi}, \varphi) \qquad \text{and} \qquad \tilde{v}(\zeta, \varphi) = \xi(-\zeta + \hat{\xi}, \varphi). \tag{4.23}$$

It now follows that

$$g_p = \ln \tilde{\xi} + \epsilon g + \epsilon^2 G + \cdots \tag{4.24}$$

and so by (4.9)

$$g = \tilde{v}\partial_{\tilde{v}} a_g(\tilde{v}, \tilde{\xi}), \tag{4.25}$$

and by (4.20)

$$G = g\tilde{\xi}\partial_{\tilde{\xi}} g + \tilde{v}\partial_{\tilde{v}}(\tfrac{1}{2} gg' + A_g) \tag{4.26}$$

with a_g and A_g antisymmetric in $(\tilde{v}, \tilde{\xi})$.

But f_p and g_p are related by a ζ translation, by (2.15)

$$g_p(\zeta, \varphi) = f_p(\zeta + \xi_g(\varphi), \varphi). \tag{2.15}$$

Here we must pause. Should ξ_g be just $\hat{\xi}$ that relates $\hat{f}$ to $\hat{g}$, or should it be modified, basically in proportion to the strength of the perturbation f? Should $\hat{\xi}$ be modified to

$$\xi_g = \hat{\xi} + \epsilon \Psi + \epsilon^2 \Phi + \cdots \tag{4.27}$$

(each term a function of φ), then certainly these fluctuations can not be freely imposed, since under a fixed pressure difference, p_g, changing $\hat{\xi}$ changes the flux of the flow, and so costs power of an amount proportional to the size of the fluctuation, and makes it possible for the external agencies driving the flow to control the fluctuations. Should Ψ, etc., however vanish, then the corresponding fluctuation is free and uncontrollable.

To be general, substitute (4.27) into (2.15) and expand in ϵ to relate g and G to f and F:

$$g(\zeta, \varphi) = (f + \hat{f}'\Psi)(\zeta + \hat{\xi}, \varphi) \tag{4.28}$$

$$G(\zeta, \varphi) = (F + \hat{f}'\Phi + \Psi \partial_\zeta (f + \tfrac{1}{2} \hat{f}'\Psi))(\zeta + \hat{\xi}, \varphi) \tag{4.29}$$

where $\partial_\zeta \Psi = 0$ since Ψ is a function just of φ.

Start with (4.28):

$$g(\zeta - \hat{\xi}, \varphi) = f + \hat{f}'\Psi.$$

Let us define for convenience

$$\Psi \equiv \lambda \psi'(\lambda). \tag{4.30}$$

Then, by (4.14)

$$\hat{f}'\Psi = -v \partial_v \lambda \psi'(\lambda) = -v \partial_v \psi(\lambda). \tag{4.31}$$

But then, by (4.9),

$$v \partial_v (a - \psi(\lambda)) = g(\zeta - \hat{\xi}, \varphi). \tag{4.32}$$

This scheme succeeds because of an extraordinary property of the (v, ξ) variables.

By (4.23),

$$\xi_* = \tilde{\xi}(\zeta - \hat{\xi}, \varphi) = \xi(\zeta, \varphi) = \xi \qquad \text{and} \qquad v_* = \tilde{v}(\zeta - \hat{\xi}, \varphi) = \xi(2\hat{\xi} - \zeta, \varphi). \tag{4.33}$$

So, by (4.11), valid just as well for $(\tilde{\xi}, \tilde{v})$,

$$\left|\frac{\partial(\ln \xi_*, \ln v_*)}{\partial(\zeta, \varphi)}\right| = \left|\frac{\partial(\ln \tilde{\xi}, \ln \tilde{v})}{\partial(\zeta, \varphi)}\right|_{(\zeta - \hat{\xi}, \varphi)} \begin{vmatrix} 1 & -\hat{\xi}' \\ 0 & 1 \end{vmatrix} = 2.$$

But then, dividing by (4.11)

$$\left|\frac{\partial(\ln \xi_*, \ln v_*)}{\partial(\ln \xi, \ln v)}\right| = 1 = \left|\frac{\partial(\ln \xi, \ln v_*)}{\partial(\ln \xi, \ln v)}\right| = \frac{\partial \ln v_*}{\partial \ln v}$$

and so,

$$v_* = \xi(2\hat{\xi} - \zeta, \varphi) = v/\gamma(\xi). \tag{4.34}$$

The remarkable result is that γ is a function purely of ξ.

Let us elaborate on (4.34). The simplicity of its deduction reflects the curious fact that the equations of motion are just the constant Jacobian of $(f(\zeta, \varphi), f(-\zeta, \varphi))$ with respect to (ζ, φ). It is straightforward to directly deduce it by the joint integration of (2.14′) for both $f(\zeta, \varphi)$ and $g(\zeta, \varphi) = f(\zeta + \xi_g, \varphi)$.

Writing

$$\gamma(\xi) = e^{\Gamma(\ln \xi)} = e^{\Gamma(f(\zeta, \varphi))}, \tag{4.35}$$

assuming (4.34) we have

$$f(-\zeta, \varphi) - f(2\xi_g - \zeta, \varphi) = \Gamma(f(\zeta, \varphi)) \tag{4.36}$$

where again, it is remarkable that Γ is a function purely of $f(\zeta, \varphi)$.

To directly deduce this result, write down (2.14′) for $g = f(\zeta + \xi_g)$ and translate the argument $\zeta \to \zeta - \xi_g$. Subtract from this (2.14′) for f and so obtain

$$0 = f'(\zeta)\, \partial_\varphi (f(-\zeta) - f(2\xi_g - \zeta)) - f_\varphi(\zeta)\, \partial_\zeta (f(-\zeta) - f(2\xi_g - \zeta)).$$

Dividing by $f' f_\varphi$, the equality of ratios is equivalent to (4.36). That is, if just f obeys (2.14′) *and* satisfies (4.36), then f is a solution for the finite problem, and conversely. This will enable us to put severe restrictions on the nature of solutions near to $\hat{f}$, a pure 1/2 finger solution; sufficiently severe to prove pattern selection.

Now, by (4.25),

$$g(\zeta, \varphi) = \tilde{v} a_{g,1}(\tilde{v}, \tilde{\xi})$$

and so,

$$g(\zeta - \hat{\xi}, \varphi) = v_* a_{g,1}(v_*, \xi_*) = \frac{v}{\gamma(\xi)} a_{g,1}\left(\frac{v}{\gamma(\xi)}, \xi\right)$$
$$= v \partial_v a_g \left(\frac{v}{\gamma(\xi)}, \xi\right). \tag{4.37}$$

But now, by (4.32),

$$a(v, \xi) - \psi(\lambda) - \rho(\xi) = a_g \left(\frac{v}{\gamma(\xi)}, \xi\right) \tag{4.38}$$

with the integration constant ρ purely a function of ξ, and we have succeeded in simultaneous integrating both equations of motion.

At this point a few observations need to be made. First λ is independent of ζ. Writing $\lambda(v, \xi)$, the value is unchanged by sending $\zeta \to -\zeta$, and so λ is symmetric in v and ξ:

$$\lambda(v, \xi) = \lambda(\xi, v). \tag{4.39}$$

(This is implicit, of course, in (4.14).) Next, write (4.34) as

$$\gamma(\xi(\zeta, \varphi)) = \frac{\xi(-\zeta, \varphi)}{\xi(2\hat{\xi} - \zeta, \varphi)}. \tag{4.40}$$

But then

$$\gamma(v_*) = \gamma\left(\frac{v}{\gamma(\xi)}\right) = \gamma(\xi(2\hat{\xi} - \zeta)) = \frac{\xi(\zeta - 2\hat{\xi})}{\xi} = \frac{v(2\hat{\xi} - \zeta)}{\xi}$$

and so, together with (4.34), we have

$$v(2\hat{\xi} - \zeta, \varphi) = \xi \gamma\left(\frac{v}{\gamma(\xi)}\right), \qquad \xi(2\hat{\xi} - \zeta, \varphi) = \frac{v}{\gamma(\xi)}. \tag{4.41}$$

But then with λ independent of ζ,

$$\lambda(\nu,\xi)=\lambda\left(\xi\gamma\left(\frac{\nu}{\gamma(\xi)}\right),\frac{\nu}{\gamma(\xi)}\right). \tag{4.42}$$

Substituting $\nu\to\nu\gamma(\xi)$,

$$\lambda(\nu\gamma(\xi),\xi)=\lambda(\xi\gamma(\nu),\nu)\equiv\Lambda(\nu,\xi):\ \Lambda^t=\Lambda \tag{4.43}$$

and Λ is another symmetric function of ν and ξ. Substituting $\nu\to\nu\gamma(\xi)$ in (4.38) now yields

$$a(\nu\gamma(\xi),\xi)-\psi(\Lambda)-\rho(\xi)=a_g(\nu,\xi). \tag{4.44}$$

Transposing (4.44) and adding, by the antisymmetry of a_g we have

$$a(\nu\gamma(\xi),\xi)+a(\xi\gamma(\nu),\nu)-2\psi(\Lambda)-\rho(\xi)-\rho(\nu)=0, \tag{4.45}$$

and we have determined what a's in (4.9) are allowed for the problem with two interfaces.

Let us see that the decomposition of (4.38) is unique, so that for a given perturbation f, and hence a given function a, both $\psi(\lambda)$ and $\rho(\xi)$ are uniquely determined. (Were this not the case then Ψ would lose meaning.) With ψ and ρ denoting the differences of two different decompositions, we have

$$-2\psi(\Lambda)=\rho(\xi)+\rho(\nu)\qquad(\text{i.e., } a=0). \tag{4.46}$$

But, with $a=0$, (4.38) is

$$a_g(\nu/\gamma,\xi)=-\psi(\lambda)-\rho(\xi),$$

and by (4.37), (4.31)

$$g(\zeta-\hat{\xi})=-\nu\partial_\nu\psi=\hat{f}'\Psi.$$

But then

$$g_p(\zeta-\hat{\xi})=\hat{f}+\epsilon\hat{f}'\Psi=\hat{f}(\zeta+\epsilon\Psi),$$

or

$$g_p=\hat{f}(\zeta+(\hat{\xi}+\epsilon\Psi),\varphi)\qquad\text{and}\qquad f_p=\hat{f}\quad(a=f=0). \tag{4.47}$$

That is, the ψ of (4.46) corresponds to the identical unperturbed motion $\hat{f}$ throughout the fluid, but we have taken a new interface at $\hat{\xi}_p = \hat{\xi} + \epsilon\Psi$. This means that we have simply considered a larger finite body of fluid, and so ψ of (4.46) is the perturbation of changing the volume of fluid. This is not what we consider, since the physical experiment is predicated on volume preservation. That is, for the physical perturbations we care about, which contain no part of ρ of the form of that which is determined by (4.46), our decomposition is unique.

At the end of Section 3, in (3.13), (3.17), and (3.38) we recorded the fluid volume through the parameter k, which is e^{2L_0} with $2\pi L_0$ the finite area of fluid. In changing L_0 by ϵ, we change k by $2\epsilon k$ and so the Ψ in (4.47) is

$$\Psi_L = k\frac{\partial(2\hat{\xi})}{\partial k}. \tag{4.48}$$

Thus we have for the flat solution by (3.13)

$$\Psi_L = 1, \qquad \psi_L = \ln\lambda \tag{4.49}$$

and for both the $\lambda = 1/2$ fingers of (3.19) and (3.38)

$$\Psi_L = \frac{k\lambda}{1+k\lambda}, \qquad \psi_L = \ln(1+k\lambda). \tag{4.50}$$

Let us now put in evidence the functions λ, γ, and Λ for these same solutions.

For (3.13),

$$\lambda = 1/\xi\nu, \qquad \gamma = e^{-2\hat{\xi}} = 1/k, \qquad \text{and} \qquad \Lambda = k/\xi\nu. \tag{4.51}$$

Similarly, for (3.17),

$$\begin{aligned} \lambda &= \frac{1}{\xi}+\frac{1}{\nu}, \\ \gamma &= \frac{\xi}{\xi+k}, \qquad \text{and} \\ \Lambda &= \frac{1}{\xi}+\frac{1}{\nu}+\frac{k}{\xi\nu}. \end{aligned} \tag{4.52}$$

Finally, for dynamical (3.38),

$$\lambda = \frac{1}{\xi} + \frac{1}{\nu} + \frac{1}{\xi\nu},$$

$$\gamma = \frac{\xi+1}{\xi+k}, \qquad \text{and} \tag{4.53}$$

$$\Lambda = \frac{1}{\xi} + \frac{1}{\nu} + \frac{k}{\xi\nu}.$$

Now, by (4.46) and (4.49), (4.50) we can determine the forbidden forms of ρ that generate pure volume changes. For (3.13)

$$\rho_L(\xi) = 2\ln\xi - \ln k, \tag{4.54}$$

and for both (3.17) and (3.38),

$$\rho_L(\xi) = -2\ln(1+k/\xi). \tag{4.55}$$

Although we have put these details in explicit evidence for the exact solutions we know of, it is to be stressed that the entire machinery of this section is directly applicable to the perturbation on *any* $\hat{f}, \hat{g}$ which solve (2.14′). In particular, it will apply to the perturbations of our explicit solutions, now regarded as new fundamental solutions, up to the order we consider.

We now need to determine $a(\nu, \xi)$ from (4.38), or eliminating a_g, from (4.45). With a antisymmetric, we can always write

$$a(\nu, \xi) \equiv u(\nu, \lambda) - u(\xi, \lambda). \tag{4.56}$$

Entering (4.56) into (4.45) and then substituting $\nu \to \nu/\gamma(\xi)$ and rearranging, we have

$$\left(u(\nu, \lambda) - u\left(\frac{\nu}{\gamma(\xi)}, \lambda\right) - \rho(\xi) - \psi(\lambda)\right)$$
$$+\left(u\left(\xi\gamma\left(\frac{\nu}{\gamma(\xi)}\right), \lambda\right) - u(\xi, \lambda) - \rho\left(\frac{\nu}{\gamma(\xi)}\right) - \psi(\lambda)\right) = 0. \tag{4.57}$$

This equation is much more transparent than it appears. Reverting to invertible (ζ, λ) variables, with

$$u(\nu, \lambda) = u(\xi(-\zeta, \lambda), \lambda) \equiv h(\zeta, \lambda), \tag{4.58}$$

(4.56) is

$$a(\zeta, \lambda) = h(\zeta, \lambda) - h(-\zeta, \lambda) \tag{4.59}$$

and (4.57), by (4.41) is

$$(h(\zeta, \lambda) - h(\zeta - 2\hat{\xi}, \lambda) - \rho(\xi(\zeta, \lambda)) - \psi(\lambda))$$
$$+ (h(2\hat{\xi} - \zeta, \lambda) - h(-\zeta, \lambda) - \rho(\xi(2\hat{\xi} - \zeta, \lambda)) - \psi(\lambda)) = 0$$

or, with the definition of $r(\zeta, \lambda)$:

$$h(\zeta, \lambda) - h(\zeta - 2\hat{\xi}, \lambda) \equiv \rho(\xi(\zeta, \lambda)) + \psi(\lambda) + r(\zeta, \lambda), \tag{4.60}$$

$$r(\zeta, \lambda) + r(2\hat{\xi} - \zeta, \lambda) = 0. \tag{4.61}$$

It is now easy to see that r can be taken to vanish with impunity. This follows because we are uninterested in h itself, but rather by (4.59), its antisymmetric part. But, (4.60) is linear in h, and so can be decomposed into three pieces of h: One piece satisfying (4.60) with $r \equiv 0$, a second with $\rho + \psi \equiv 0$ with r satisfying (4.61), and finally a purely homogeneous part

$$h_0(\zeta, \lambda) = h_0(\zeta - 2\hat{\xi}, \lambda)$$

and so with

$$a_0(\zeta, \lambda) = a_0(\zeta - 2\hat{\xi}, \lambda). \tag{4.62}$$

As we have already discussed, an antisymmetric doubly periodic function must have singularities within physical fluid, so that a_0 can be dropped by simply determining the rest of h to have correct analyticity. (This is correct, but not quite the entire argument, since it is not a's singularity in fluid, but f's, its $\ln v$ derivative, that matters. See the paragraph following (4.75).) Returning to the second r contributed piece of h, h_r, and suppressing the λ-dependence,

$$h_r(\zeta) - h_r(\zeta - 2\hat{\xi}) = r(\zeta).$$

Setting $\zeta \to 2\hat{\xi} - \zeta$, and using (4.61),

$$h_r(-\zeta) - h_r(2\hat{\xi} - \zeta) = r(\zeta) = h_r(\zeta) - h_r(\zeta - 2\hat{\xi}),$$

or

$$a_r(\zeta, \lambda) = a_r(\zeta - 2\hat{\xi}, \lambda)$$

which then can be totally absorbed into a_0, and then a_0 dropped completely for an h with correct analyticity, which then simply obeys

$$h(\zeta, \lambda) - h(\zeta - 2\hat{\xi}, \lambda) = \rho(\xi(\zeta, \lambda)) + \psi(\lambda).$$

For a convenience of geometry (what we refer to mentally as $\operatorname{Re} \zeta \to +\infty$), it is convenient to redefine $h(\zeta, \lambda) \to -h(-\zeta, \lambda)$, which leaves (4.56) and (4.59) unchanged, and produces, after sending $\zeta \to -\zeta$

$$h(\zeta + 2\hat{\xi}, \lambda) - h(\zeta, \lambda) = \rho(v(\zeta, \lambda)) + \psi(\lambda). \tag{4.63}$$

The unique solution of (4.63)—the one which possesses correct annular analyticity—is then the unique solution to (4.45).

Let us next note that for the exact solutions of (4.51)–(4.53), as $\operatorname{Re} \zeta \to +\infty$, v vanishes for the flat solution and has, for the other two $1/2$ finger solutions

$$v \to 1/\lambda \qquad \text{as} \quad \operatorname{Re} \zeta \to +\infty, \tag{4.64}$$

so that solutions to (4.63) with h regular as $\operatorname{Re} \zeta \to +\infty$,

$$\psi(\lambda) = -\rho(1/\lambda), \tag{4.65}$$

for the $1/2$ finger solutions, but

$$\psi(\lambda) = -\rho(0) \Rightarrow \Psi = 0 \tag{4.66}$$

for the flat solution. For the analogs of the infinite fluid problem, with $\rho = v^n/n$, $\Psi = \lambda\psi'(\lambda) = e^{2n\varphi}$, and so nothing is changed for the flat solution's perturbations, but now for perturbations about the $1/2$ finger, impedance (flux) grows exponentially in proportion to the instability's strength, and so are not free, and controllable by the pump that drives the flow: It is exactly as potent to control the change of flux as to control the growth of these instabilities. This is totally different from the infinite fluid case where these instabilities were freely impressible and uncontrollable. At this point we already mathematically conclude that the geometry of infinite fluid is physically incorrect, and the limit of large length clearly singular. This is the first main point we set out to establish.

As for h's that are regular as $\operatorname{Re}\zeta \to -\infty$, all v's of the exact solutions diverge, and so the ρ's that determine them have constant limits (for simple powers, zero), and so these all have $\Psi = 0$. These will all turn out, as do the powers which relax as $e^{-2n\varphi}$, as relaxing stabilities. Thus, the flat solution is uncontrollably unstable, but the 1/2 finger solution can surely, under appropriate experimental control, be reduced to non-exponential slow modes. It is our goal to show that these putative slow modes are non-existent.

At this point we drop any further focus upon the flat solution, and restrict ourselves to the dynamical 1/2 finger of (4.53), since (4.52) is simply a long-time, long-fluid limit of this. We will now write down all possible "perturbations" f upon this solution.

Denote by x, y:

$$y \equiv e^{\zeta - \hat{\zeta}(\lambda)}, \qquad x \equiv y(-\zeta, \lambda) \Rightarrow xy = e^{-2\hat{\zeta}} = 1 + \lambda \tag{4.67}$$

so that

$$v = (1+x)/\lambda, \qquad \xi = (1+y)/\lambda. \tag{4.68}$$

Denote by $\beta(\lambda)$:

$$\beta(\lambda) \equiv e^{-2\hat{\xi}} = \frac{1+\lambda}{1+k\lambda} \tag{4.69}$$

by (3.38). Finally, define

$$v(x, \lambda) \equiv h(\zeta, \lambda), \tag{4.70}$$

so that (4.63) reads

$$v(\beta x, \lambda) - v(x, \lambda) = \rho\left(\frac{1+x}{\lambda}\right) + \psi(\lambda), \tag{4.71}$$

and

$$a(v, \xi) = v(x, \lambda) - v(y, \lambda). \tag{4.72}$$

By straightforward differentiation, suppressing the λ-dependence,

$$f = (1+x)\, v'(x) - \frac{y}{1+y}\left((1+x)\, v'(x) + \lambda v_\lambda(x) - (x \to y)\right). \tag{4.73}$$

The flux-bearing solutions, $x \to 0$, are, by summing (4.71) at the contractive ($\beta < 1$) fixed point at 0,

$$\begin{aligned} v_d(x,\lambda) &= -\sum_0 \left(\rho\left(\frac{1+\beta^n x}{\lambda}\right) - \rho\left(\frac{1}{\lambda}\right)\right); \\ \psi &= -\rho\left(\frac{1}{\lambda}\right) \Rightarrow \Psi = \frac{1}{\lambda}\rho'\left(\frac{1}{\lambda}\right) \end{aligned} \tag{4.74}$$

which we call "down-summing," or "$d\Sigma$."

The flux-free stabilities, $x \to \infty$, are, by summing the inverse,

$$v_u(x,\lambda) = -\sum_1 \left(\rho\left(\frac{1+\beta^{-n} x}{\lambda}\right) - \rho(\infty)\right), \qquad \Psi = 0 \tag{4.75}$$

which we call "up-summing," or "$u\Sigma$." (It is immediate to verify that the ζ derivative of $v_d + v_u$ is precisely doubly-periodic when both sums exist.) Should ρ be neither up nor down summable, then we simply split it into $\rho = \rho_u + \rho_d - \psi_\rho(\lambda)$ with ψ_ρ, a piece of the ψ for this ρ. (This is just a Laurent expansion when not explicitly possible.) Thus, we can determine all possible v's.

Let us return to the non-existence of the homogeneous solutions, a_0 of (4.62). With a term that is singular within physical fluid, but of the precise form $u(\xi)$, f_0 will be acceptable, but not otherwise. Such a doubly-periodic a_0 then must have the form

$$a_0 = \sum_{-\infty}^{+\infty} \left(u\left(\frac{\lambda}{1+\beta^n x}\right) - u\left(\frac{\lambda}{1+\beta^n y}\right)\right)$$

where, for convergence, $u(\lambda)$ is analytic on some interval $[0, \lambda_M]$ of the real axis. To be doubly-periodic, $0 = a_0(\zeta + 2\hat{\xi}, \lambda) - a_0(\zeta, \lambda)$. The first term is obtained by replacing x by βx and y by $\beta^{-1} y$. Upon subtracting and taking the limits to infinity, obtain $0 = 2(u(\lambda) - u(0))$. Then, by analytic continuation, such a u is identically constant, and so a_0 vanishes. Thus, there are no homogeneous solutions at all.

Should $\rho = 0$ in (4.63), then unless $\psi = \hat{\xi}$, the corresponding f has a term linear in ζ, which violates channel geometry. If the condition is met, it together with f are the generators of φ-translations, the underlying symmetry of the theory. That is, all perturbations are in one-to-one correspondence with ρ's.

Let us now write

$$R(v) \equiv -v\rho'(v). \tag{4.76}$$

For $d\Sigma$ solutions, by (4.74),

$$\Psi(\lambda) = -R(1/\lambda). \tag{4.77}$$

To determine f, we need one last object, which by (4.69) is

$$-\lambda \frac{\beta'}{\beta} \equiv \alpha(\lambda) = \frac{1-\beta}{1+\lambda}. \tag{4.78}$$

Then, by (4.73),

$$\begin{aligned} f_d = R(v) + \sum_1 \frac{(1+x)\,\beta^n}{1+\beta^n x} R\left(\frac{1+\beta^n x}{\lambda}\right) \\ + \frac{y}{1+y} \sum_1 \left(\frac{1-\beta^n + nx\beta^n \alpha(\lambda)}{1+\beta^n x} R\left(\frac{1+\beta^n x}{\lambda}\right) - R\left(\frac{1}{\lambda}\right)\right) - \frac{y}{1+y} \sum_1 (x \to y). \end{aligned} \tag{4.79}$$

This equation, in conjunction with (4.77) is remarkable: It says that the pump's measurement of extra requested flux as a function of time uniquely determines and is uniquely determined by the *entire* spatial shape of the flow: The pump can literally be programmed to create or reciprocally control any allowed unstable conformation of the fluid!

Next, in the finger regime $\beta \sim 1/k\lambda$ and is astronomically, exponentially small, so that the sums in (4.79) are astronomically small, so that $f_d \sim R(v)$, just the result of the infinite fluid with one interface. For example, $R = \ln(1+av)$ produces a pole-dynamics singularity

$$R = \ln\left(1 + \frac{a}{\lambda}(1+e^{-\zeta-\hat{\zeta}})\right) = \ln\left(1+\frac{a}{\lambda}\right) + \ln(1-e^{\zeta_s-\zeta}) \tag{4.80}$$

with

$$e^{\zeta_s} = \frac{-e^{-\hat{\zeta}}}{1+\lambda/a}$$

which for $a \in (0, 2)$ is always $\zeta_s < 0$ for all λ, and for $a = 1$ is just $\zeta_s = \hat{\zeta}$. Clearly, as $\beta \to 0$ for a long enough channel during the finger regime, we have then a propagating finger of width $(1-\epsilon)/2$, and all fingers are allowed. So it would seem. (It is false.)

In any case, with $R(v)$ analytic as $\operatorname{Re}\zeta \to +\infty$, these down sums are perturbations exponentially stronger (by roughly $e^{\hat{\zeta}} \sim e^{L}$) on the driven interface than on the efflux interface, and clearly by (4.77) all instabilities.

Finally, let us notice where all of f_d's singularities are. First, $R(0)=0$ since by (4.76) this is true unless $\rho=\ln v+\cdots$ which by (4.55) means it contains a piece that changes the area of the fluid, which is illegal. Thus the poles at $1+\beta^n x=0$ and $1+\beta^n y=0$ are removable. The pole at $y+1=0$, (or $\xi=0$) is a higher order correction to the basic singularity of $\hat{f}=\ln\xi$, and is of no special concern. Since $\beta^n=e^{-2n\hat{\xi}}$, we then see that f_d of (4.79) has singularities at

$$\zeta_{+n}=\zeta_s-2n\hat{\xi} \quad n=0\ldots \qquad \text{and} \qquad \zeta_{-n}=2n\hat{\xi}-\zeta_s \quad n=1\ldots \tag{4.81}$$

when $R(v)$ has a singularity at $\zeta_0=\zeta_s$. Most important is that $\zeta_{-0}=-\zeta_s$ is *not* a singularity. (By (4.74) the 0 term in the sum yields $a=\rho(\xi)-\rho(v)$ and $f=v\partial_v a$ dispatches the $\rho(\xi)$ term which contains the putative singularity at $-\zeta_s$.) In consequence $-\zeta_s$ can lie in physical fluid $(0,\hat{\xi})$, and so, f_d's can have singularities below, but arbitrarily close to $\operatorname{Re}\zeta=0$, and so able to significantly modify the shape of the driven interface. Again, so it would seem.

Also, in particular, f_d is singular at $2\hat{\xi}-\zeta_s$ if it is at ζ_s. This will prove to be of the utmost consequence.

The up-sum solutions are of a totally different character. By (4.75) we have

$$f_u=-\sum_1\frac{1+x}{\beta^n+x}R\left(\frac{1+\beta^{-n}x}{\lambda}\right)$$
$$+\frac{y}{1+y}\sum_1\frac{1-\beta^n+nx\alpha(\lambda)}{\beta^n+x}R\left(\frac{1+\beta^{-n}x}{\lambda}\right)-\frac{y}{1+y}\sum_1(x\to y). \tag{4.82}$$

It is useful to note that the sum of the two $n=1$ terms in x produces

$$f_u=-R\left(\frac{1+\beta^{-1}x}{\lambda}\right)+\cdots, \tag{4.83}$$

and f_u in the finger regime is well approximated by just this leading term. As for singularities, with $R(v)$ possessing a singularity at ζ_s, f_u is singular at

$$\zeta_{\pm n}=\pm(\zeta_s+2n\hat{\xi}) \qquad n=1,2,\ldots \tag{4.84}$$

However, apart from ζ_{+1}, by (4.78) all other terms are also singular at

$$\zeta_{\pm n}=\pm(-\hat{\zeta}+2n\hat{\xi}) \quad n=1,2,\ldots \qquad \text{but not } \zeta_{+1}. \tag{4.85}$$

Should R be logarithmic, (i.e., have cuts) such as (4.80), then the first term, (4.83) is

$$\ln(1 - e^{\zeta_s + 2\hat{\xi} - \zeta}). \tag{4.86}$$

This has two immediate consequences. First, $\zeta_s > -\hat{\xi}$ so that $\zeta_s + 2\hat{\xi} > \hat{\xi}$, and so outside of physical fluid. (For ζ_s more negative than $-2\hat{\xi}$, one of the ζ_{+n} will still be in fluid.) Thus, such perturbations can only be imposed after a time with λ sufficiently small. Second, with $\zeta_s + 2\hat{\xi} > \hat{\xi}$, the only branching of (4.86) that is reflection symmetric is with the cut running off to the left, and so throughout physical fluid, thereby violating channel width. Accordingly, branched $R(v)$'s must contain paired singularities, with cuts beginning on one singularity, and terminating on a mating one (or ones). That is, if

$$R = \Sigma \alpha_n \ln(1 + a_n v),$$

then

$$\Sigma \alpha = 0. \tag{4.87}$$

With (4.87) enforced, and with R logarithms, the long fluid behavior of (4.83) is that of pole-dynamics with singularities to the right of fluid (low pressure) sinks, but sinking zero net flux.

For both second order perturbation and the theory of perturbations about perturbed solutions, we require some facts about first order solutions which follow by differentiating (4.43) and (4.45). Differentiating (4.43) by $\xi \partial_\xi$, and setting $v \to v/\gamma$ yields by (4.14) and (4.41)

$$\frac{\xi \gamma'}{\gamma} \hat{f}'(\zeta) + \hat{f}'(-\zeta) = \hat{f}'(2\hat{\xi} - \zeta), \tag{4.88}$$

a result just as well obtained by differentiating ln(4.40) on ζ.

Differentiating (4.45) by $\xi \partial_\xi$, together with a's antisymmetry produces, after noticing that

$$\frac{1}{\Lambda} \xi \partial_\xi \Lambda = \frac{1}{\Lambda} \xi \partial_\xi \lambda(\xi \gamma(v), v) = -\hat{f}'(\xi \gamma(v), v)$$

and setting $v \to v/\gamma(\xi)$,

$$\frac{\xi \gamma'}{\gamma} f(\zeta) - f(-\zeta) + f(2\hat{\xi} - \zeta) + 2\Psi(\lambda)\, \hat{f}'(2\hat{\xi} - \zeta) = \xi \rho'(\xi) \equiv -R(\xi). \tag{4.89}$$

As a last observation important for second order theory, by (4.14),

$$P \equiv \Psi(\lambda)\, \hat{f}' = -\nu \partial_\nu \psi \qquad \text{and} \qquad P^t \equiv \Psi(\lambda)\, \hat{f}'^t = -\xi \partial_\xi \psi \tag{4.90}$$

so that

$$\xi \partial_\xi P = \nu \partial_\nu P^t. \tag{4.91}$$

Notice also that

$$\tilde{\nu} \partial_{\tilde{\nu}} u|_{\zeta - \hat{\xi}} = \nu \partial_\nu u \left(\frac{\nu}{\gamma(\xi)}, \xi \right)$$

and

$$\tilde{\xi} \partial_{\tilde{\xi}} u|_{\zeta - \hat{\xi}} = \left(\xi \partial_\xi + \frac{\xi \gamma'}{\gamma} \nu \partial_\nu \right) u \left(\frac{\nu}{\gamma(\xi)}, \xi \right).$$

It is now straightforward to write out second order perturbation theory. First calculate the pull-back of G of (4.26) to $\zeta - \hat{\xi}$. By (4.28)

$$g(\zeta - \hat{\xi}) = g\left(\frac{\nu}{\gamma}, \xi \right) = (f + P)(\nu, \xi), \qquad \text{or}$$

$$g(\nu, \xi) = (f + P)(\nu\gamma(\xi), \xi).$$

For $g^t = g(\tilde{\xi}, \tilde{\nu})$, we have

$$g^t(\zeta - \hat{\xi}) = g\left(\xi, \frac{\nu}{\gamma} \right) = (f + P)\left(\xi\gamma\left(\frac{\nu}{\gamma} \right), \frac{\nu}{\gamma} \right) = (f + P)(2\hat{\xi} - \zeta).$$

Now, using (4.89) and (4.88) multiplied by Ψ, obtain

$$\frac{1}{2} g g^t(\zeta - \hat{\xi}) = \frac{1}{2}(f + P)(f^t - P^t) - \frac{\xi\gamma'}{2\gamma}(f + P)^2 - \frac{1}{2}(f + P)\, R(\xi).$$

Combined with the first term of (4.26) pulled back to $\zeta - \hat{\xi}$, we have

$$\begin{aligned} G(\zeta - \hat{\xi}) &= (f + P)(\xi \partial_\xi f + \nu \partial_\nu P^t) \\ &\quad + \nu \partial_\nu \left(\frac{1}{2}(f + P)(f^t - P^t - R(\xi)) + A_g\left(\frac{\nu}{\gamma}, \xi \right) \right). \end{aligned}$$

We now equate this expression to $G(\zeta-\hat{\xi})$ as obtained from f_p, (4.29), using (4.15) multiplied by Ψ. Defining the second order flux

$$\Phi \equiv \lambda\varphi'(\lambda), \tag{4.92}$$

after obvious cancellations, we produce an expression of the form $v\partial_v(\ )$, and so, with an integration constant $\mu(\xi)$ purely of ξ, obtain the fully integrated second order analog to (4.38)

$$\begin{aligned} &A(v, \xi)-\varphi(\lambda)-\mu(\xi) \\ &\quad=\frac{1}{2} f(P^t-R(\xi))+\frac{1}{2} P(f^t-R(\xi))+A_g\left(\frac{v}{\gamma}, \xi\right). \end{aligned} \tag{4.93}$$

We now proceed exactly as we did to achieve (4.60), with now an extra right-hand driving term. φ and μ are chosen as the minimal required terms to produce an A analytic over physical fluid. Here we must generally avail ourselves of an appropriate doubly periodic homogeneous solution, should the particular solution determined by the driving term from first order happen to produce unacceptable singularities. (The form of (4.93) with $f^t-R(\xi)$, and the construction of F from the potential $A+\frac{1}{2}ff^t$, however, eliminates such problems.)

To end this section on formalities, let us see how γ, or Γ is modified to first order in ϵ from $\hat{\gamma}$ and $\hat{\Gamma}$. Expanding (4.36),

$$\begin{aligned} &\hat{f}(-\zeta)-\hat{f}(2\hat{\xi}-\zeta)-2\epsilon\Psi\hat{f}'(2\hat{\xi}-\zeta)+\epsilon(f(-\zeta)-f(2\hat{\xi}-\zeta)) \\ &\quad=\Gamma(\hat{f}(\zeta)+\epsilon f(\zeta))+O(\epsilon^2). \end{aligned} \tag{4.94}$$

With $\hat{f}(-\zeta)-\hat{f}(2\hat{\xi}-\zeta)=\hat{\Gamma}(\hat{f}(\zeta))$, and dividing by ϵ,

$$f(-\zeta)-f(2\hat{\xi}-\zeta)=\frac{1}{\epsilon}(\Gamma(\hat{f}(\zeta)+\epsilon f(\zeta))-\hat{\Gamma}(\hat{f}(\zeta)))+2\Psi\hat{f}'(2\hat{\xi}-\zeta)+O(\epsilon).$$

Since, by (4.35),

$$\hat{\Gamma}'(\hat{f}(\zeta))=\xi\frac{\gamma'}{\gamma}(\xi), \tag{4.95}$$

by (4.89), the perturbation theory,

$$\frac{1}{\epsilon}(\Gamma(\hat{f}(\zeta)+\epsilon f(\zeta))-\hat{\Gamma}(\hat{f}(\zeta)))=R(e^{\hat{f}})+\hat{\Gamma}'(\hat{f}(\zeta))\,f(\zeta)+O(\epsilon).$$

Taking the limit $\epsilon \sim 0$ (Γ must be analytic),

$$\Gamma(\hat{f}) = \hat{\Gamma}(\hat{f}) + \epsilon R(e^{\hat{f}}) + O(\epsilon^2).$$

That is, for f_p to be the full perturbed solution to our problem, and with $\xi = e^{f_p(\zeta)}$,

$$\Gamma(f_p(\zeta)) = \hat{\Gamma}(f_p(\zeta)) + \epsilon R(\xi) + O(\epsilon^2) \tag{4.96}$$

or,

$$\gamma(\xi) = \hat{\gamma}(\xi)(1 + \epsilon R(\xi)) + O(\epsilon^2). \tag{4.97}$$

(Indeed, after significant cancellations, there results to second order that

$$\Gamma(f_p(\zeta)) = \hat{\Gamma}(f_p(\zeta)) + \epsilon R(\xi) + \epsilon^2 R_2(\xi) + O(\epsilon^3), \tag{4.98}$$

where R_2 is obtained from μ of (4.93) just as was R from ρ. Generally, there is no need to add a new excitation in second order, so that we can take R_2 to vanish, so that (4.96) is uncorrected at least through second order.) This result has deep consequences for perturbations, which we now spell out.

5. DERIVATIVES VERSUS PERTURBATIONS

The idea of perturbation theory is that given a solution, we find all the (partial in ϵ) derivatives of the equations of motion about this point which, in principle, determines a power series for the solution with leading term the known solution. Should the solution actually be differentiable over some compact region, then the first order solution $\hat{f} + \epsilon f$, (f just a particular tangent) with ϵ small but finite, agrees with an actual solution over the region within a uniform $O(\epsilon^2)$ error. In general one needs either powerful bounds or the entire series to all orders with the knowledge of its convergence, to know if this derivative actually exists. It is only then that $\hat{f} + \epsilon f$ is an approximation to a solution. And this can almost never be settled to any finite order in perturbation without exceptional extra knowledge. The exact result (4.36) and it uniquely determined perturbed form, (4.96) provides us with some such knowledge.

Specifically, (4.97) says that if $\hat{f} + \epsilon f$ for a finite, fixed value of ϵ is an approximation to an actual solution, then γ exists as a function purely of ξ, and the leading term and part of its $O(\epsilon)$ corrections resides in $\hat{\gamma}(\xi)$. That is, in (4.96) what must be true is that the first term is

$$\hat{\Gamma}(\hat{f} + \epsilon f) \tag{5.1}$$

and *not*

$$\hat{\Gamma}(\hat{f})+\epsilon\hat{\Gamma}'(\hat{f})\, f+O(\epsilon^2). \tag{5.2}$$

Evidently, if f is always bounded over the region we care about, then (5.2) is altogether equivalent to (5.1). Should f not be bounded over the region of control then only (5.1) applies, and (5.2) is wrong. The *formal* $O(\epsilon)$ perturbative result, (4.89), relies upon (5.2), not (5.1). That is, if the perturbation f entails singularities there are deep potential problems unless ϵ is viewed as an actual infinitesimal.

Now, what we might have liked to say is that the region of concern is just $[0, \hat{\xi}]$, the domain of physical fluid, so that even if f has singularities outside this region, we need pay attention only to a domain free of them in which (5.1) and (5.2) are equivalent. That *both* interfaces must be satisfied, with its specific form of the method of images, (4.36) prevents us from determining satisfaction just within $[0, \hat{\xi}]$. By the equations of motion for the driven interface, singularities within $[-\hat{\xi}, 0]$ must be compatible (e.g., have residues correctly determined to $O(\epsilon)$) with the analytic form of f_p within $[0, \hat{\xi}]$. But also, the simultaneous satisfaction of the g equation, yielding (4.36) requires including the behavior of f up to a least $2\hat{\xi}$, if not even $3\hat{\xi}$ (since $-\zeta$, near a singularity, can approach $\hat{\xi}$). Thus, $f_p \simeq \hat{f}+\epsilon f$ must consistently agree with singularity structure over a region basically $[-\hat{\xi}, 3\hat{\xi}]$. Indeed there is every reason to expect $\hat{f}+\epsilon f$ to arbitrarily disagree outside this domain, so far beyond physical fluid.

Let us first see that the up-summed solution, stabilities with their far-away singularities, is in conformity with (4.36) and (5.1) throughout this domain, even at singularities. Within this domain, only the leading term (4.83) possesses a singularity, at $\zeta_s+2\hat{\xi}$ when $R(v)$ has a singularity at ζ_s. Since in this case $\zeta_s \in [-\hat{\xi}, 0]$ in the worst case (strongest perturbation), f_p must be analytic at $-\zeta_s$, which is then in physical fluid. So, in (4.36) we consider

$$\begin{aligned} \zeta &= -\zeta_s + x, \\ f_p(\zeta_s - x) - f_p(2\xi_g + \zeta_s - x) &\sim \Gamma(f_p(-\zeta_s + x)) \\ &\sim \hat{\Gamma}(f_p(-\zeta_s + x)) + \epsilon R(e^{\hat{f}(-\zeta_s + x)}). \end{aligned} \tag{5.3}$$

Now, f_p's perturbing part is singular at $2\xi_g+\zeta_s$ by (4.83), f_p is regular at $\pm\zeta_s$, as is $\hat{\Gamma}(f_p(-\zeta_s)))$, but R is also singular at $-\zeta_s$. Indeed, (4.83) of perturbation theory exactly agrees with this, and so, we conclude that up-summing, even with singularities, is compatible with f_p existing as an approximate solution.

Next we turn to down-summing. With $R(v) = v^n$, $n > 0$, it is elementary to exactly sum (4.79), while by (4.77), such a perturbation is flux-bearing with $\Psi_n = -\lambda^{-n} = -e^{2n\varphi}$, rapidly exponentially growing, but visible to the pump driving the flow, and so in principle, controllable. These perturbations are nonsingular, and so for small enough ϵ (5.1) and (5.2) are equivalent. At the end-points of the domain we must consider, the perturbing f's become enormous, so that $\hat{f} + \epsilon f$ is probably reliable only for $\epsilon \sim e^{-L}$. For larger ϵ, when the perturbation has grown significantly with φ, we have no idea what its nonlinear fate is.

Finally, we consider summing the R_n's into singularities, to produce, for example, pole-dynamical R's such as $\ln(1+av)$. Let us now see that if these singularities lie within $[-\hat{\xi}, 0]$, then the perturbative result is not an approximation to any solution of the problem. The test is for

$$\zeta = \zeta_s + x$$

where, for $\zeta_s \in [-\hat{\xi}, 0]$, $-\zeta_s$ is within physical fluid. Here, (4.36) is

$$f_p(-\zeta_s - x) - f_p(2\xi_g - \zeta_s - x) = \Gamma(f_p(\zeta_s + x)),$$

or, with (4.96), and picking off singular terms,

$$-\epsilon f_d(2\hat{\xi} - \zeta_s - x) \sim \hat{\Gamma}(\hat{f}(\zeta_s) + \epsilon f_d(\zeta_s + x)) + \epsilon R(\hat{f}(\zeta_s)). \qquad (5.4)$$

Now, R is regular at $\hat{f}(\zeta_s)$ (it's singular at $\hat{f}(-\zeta_s)$), but $\hat{\Gamma}$ is to be evaluated at an infinite value of its argument, at which, in fact, it is regular. However, by (4.79), $2\hat{\xi} - \zeta_s$ *is* a singularity of f_d, and indeed, within the domain of concern, only ζ_s and $2\hat{\xi} - \zeta_s$ are singularities. But then, (5.4) is inconsistent. (Had we used the incorrect form, (5.2), the two sides would have agreed with identical residues.) *That is, down-summed solutions are compatible with being approximations provided that* $R(v)$ *is free of singularities in* $[-\hat{\xi}, 0]$. This implies isolated pattern selection. To produce a finger of different width, the leading pole-dynamic term $R(v) = \ln(1+av)$ is required for values of a not exponentially small with fluid length L. In this case during the finger regime $\lambda \ll 1$, $k\lambda \gg 1$, or $1 \ll \varphi \ll L$, the singularity ζ_s must move $O(e^{-2\varphi})$ near to $\zeta = 0$ for the logarithmic stretching to take place that modifies finger width. To say that $\hat{f} + \epsilon R$ for fixed finite ϵ is an approximate solution is that

$$f_p \simeq \hat{f} + \epsilon \ln(1 + a/\lambda) + \epsilon \ln(1 - e^{\zeta_s - \zeta}),$$

so that near ζ_s,

$$f'_p \sim \frac{\epsilon}{e^{\zeta-\zeta_s}-1}, \qquad f_{p,\varphi} \sim \frac{-\epsilon\zeta'_s}{e^{\zeta-\zeta_s}-1}$$

so that

$$\frac{f_{p,\varphi}}{f'_p}\,\overrightarrow{\zeta \to \zeta_s}\, -\zeta'_s.$$

Dividing the equation of motion for the driven interface by $f'_p(\zeta)$, we have, just as for pole dynamics

$$f_p(-\zeta_s(\varphi), \varphi) = x_s + O(\epsilon^2), \qquad x'_s = 0.$$

That is, if $\ln(1+av)$ is to produce a changed finger, f_p near $\mathrm{Re}\,\zeta = 0$ must include the singularity and $f_p(-\zeta_s)$ must produce a stagnation point. f_p near its other nearest singularity, presumably at $2\hat{\xi} - \zeta_s$ must be compatible with the method of images (4.36). That is, (4.36) must be satisfied for $|\mathrm{Re}\,\zeta| < \delta$ with $\delta \to 0^+$. This is exactly what we have demonstrated is false in (5.4).

Let us emphasize that we are not saying that there can be no such solutions, nor are we saying that the translating solution is stable and isolated. We are simply saying that such solutions are not smoothly connected to the unperturbed solution. In order to see these solutions, it is first necessary that the translating solution has become sufficiently distorted so as to no longer be nearby. Near enough to this solution, there is no other with a well-defined shape and slow time variation of impedance. Rather, there is just the $1/2$ finger and exponential growths away from it. Under impedance control, we can maintain the solution within this nearby area, and so, in just this sense, determine a unique pattern.

Indeed, with singularities never within $[-\hat{\xi}, 0]$, the perturbation is exponentially small with fluid length during the finger regime, and so for a long enough body of fluid, without impact on the solution $\hat{f}$. That is, we have ruled out the possibility raised at the beginning of Section 4 that requesting an exactly translating solution might be too brittle a question to determine selection for a finite body of fluid. Furthermore, with $\hat{f}$ warding off singularities in $[-\hat{\xi}, 0]$, or roughly by the length of physical fluid, the greatest host of perturbations that could induce finite-time singularities have also been warded off. All that is significantly left are the exponentially growing v^n perturbations. Left to themselves, these rapidly lead $\hat{f}$ into a nonlinear regime about which we know little, but surmise tip-splitting. On the other hand, as they proportionately modify flux, they are controllable. It is conceivable that a simulation of this flow constrained to prevent rapid flux variations might, near enough to $\hat{f}$, be stable.

Apart from these exponentially growing fluxes, there are no slow modes to modify $\hat{\xi} \sim L_0 - \varphi$ to $\hat{\xi} \sim L_0 - (1/\lambda - 1)\,\varphi$. That is, this flow protects the far downstream behavior of the $1/2$ finger without surface tension on the efflux. The ability of $\hat{f}$ to ward off nearby singularities lies in its ability to satisfy (4.36) *without* any image singularities present, whereas all putative nearby perturbations with singularities must possess them, and are then inconsistent. Clearly, with surface tension present, a method of images reminiscent of (4.36) must exist, and so one can surmise that the appropriate S–T solution is again special. Even though surface tension is a perturbation that spoils the critical symmetry of this paper, there is now a reinforced reason to surmise that (3.31),

$$1 - 2\lambda = (2\pi)^2\, B_e,$$

at least for λ near $1/2$ is correct, which I offer as a conjecture.

6. DISCUSSION

We have followed a clue laid down by reflection symmetry that the analytically continued equations of motion are symmetric under parity, $\zeta \to -\zeta$. This suggests that far upstream ($\mathrm{Re}\,\zeta < 0$) singularities, which determine the shape of the driven interface at $\mathrm{Re}\,\zeta = 0$, are related to and co-determined with far downstream behavior, which is the nature of the efflux in a physical experiment. To see if any such thing holds, it was then mandatory to consider the flow of a finite body of fluid. The simplest feasible such problem, fully posed within the purview of the analytically continued conformal machinery, consists in replacing the Riemann mapping of a disc to an annulus. The case of the simpler disc embraces all prior studies, and corresponds to fluid going off infinitely far to the right with a simple pole at $+\infty$. In the annular version, the available channel is still infinite, but fluid terminates at a downstream interface, the behavior of which constitutes the efflux for this configuration.

Requesting a purely translating driven interface receives a unique solution in the finite problem: There is sharp, precise selection for this question. Examined in pole-dynamics, even including singularities to the right of the efflux interface, precisely and only this selected solution exists. Examining a class of solutions that exist for the problem of the disc, but not here; namely the usual Saffman–Taylor solutions, we notice that selection is expressed on the efflux interface, even when asymptotically far downstream, with only the $1/2$ solution maintaining it at a fixed pressure. Solutions with $\lambda < 1/2$ also correspond to solutions to the finite problem, but with surface tension now smoothing the efflux interface (although, of course, with zero surface tension still on the driven interface). The explicit relationship, $1-2\lambda = \sigma/v$ provides the physical interpretation of all the S–T solutions. Moreover, that the infinite fluid version of the disc allows all these possibilities with equanimity, determines that its pole at $+\infty$ is truly within physical fluid, so that the problem of the disc is the theory of an unfinished, unterminated experiment, and is not the limit of a very long body of fluid. That is, the physical limit $L \to \infty$ is singular.

We end up showing that solutions to the finite problem near the selected one all preserve the efflux condition $1-2\lambda = 0 = \sigma/v$. Although we have not worked out the theory for surface tension on the efflux, considering that the $\lambda \neq 1/2$ solutions do meet this condition, and that the condition is conserved under perturbations in the $\lambda = 1/2$ case, it is the most natural surmise that the width-efflux condition is generally correct, at least for λ not too much below $1/2$.

Most importantly, we reach the conclusion that while many consequences of the distant efflux are exponentially small, this is false for pattern selection. This is so different from the generally presumed behavior that for this reason alone, the annular configuration we discuss should be put to the experimental test. It is important in this regard to note (private communications from several of the experimenters) that this selection with beautifully formed fingers is not so easy to experimentally produce, generally requiring fiddling with the efflux to become reliably reproducible. For example, Tabeling *et al.*[3] succeeded only after fitting an "impedance-matching" plug to the efflux. It is regrettable that the experimenters haven't carefully observed and calibrated such dependencies.

Having determined that at least for the elementary S–T solutions that only $\lambda = 1/2$ has any connection to a physically finite flow with efflux at atmospheric pressure and no surface tension on it, we go on to question if asking for pure translation in a finite problem is perhaps too brittle a request. In the flow of a finite piece of fluid, there is a finite, definite period of time over which the solution is that of a finger with a large body of fluid still downstream from it. The driven interface at velocity $1/\lambda$ penetrates into the fluid whose terminating interface, virtually flat, moves with velocity 1. For $\lambda = 1/2$, by the time the fluid has translated its resting length L under the mean velocity of its flux (i.e., 1), there remains but a thin layer of fluid still downstream from the driven interface, as the efflux interface begins to curve into a similar finger. Calling this finite period of time when the pure finger is well-formed (i.e., with long horizontal sides) but still with a large body of fluid behind it the "finger regime" (as opposed to afterwards, the "breakthrough regime"), we can ask if there are solutions that within measurable error appear as perfectly translating, but actually possess deviations exponentially small with L, only becoming physically important in the breakthrough regime. Should this be the case, then the strong selection is a misleading artifact of too mathematically posed a question.

We determine the nature of solutions near the pure $1/2$ S–T finger, through a perturbative analysis. As we erect this machinery, we are immediately led to notice that exceptionally useful new variables, basically $(f(\zeta, \varphi), f(-\zeta, \varphi))$, constructed from the old (ζ, φ), are guaranteed of invertiblity, since the equation of motion for the driven interface is precisely the statement that the Jacobian is the constant 2. Translating in ζ an identical fact for the equation of motion for the efflux interface, we notice that given the first equation of motion, the second can be fully integrated into the relationship

$$f(-\zeta, \varphi) - f(2\xi_g - \zeta, \varphi) = \Gamma(f(\zeta, \varphi)). \tag{Γ}$$

What is crucial here is that Γ depends *only* upon the one new variable $f(\zeta, \varphi)$. This formula serves as the precise way in which the method of images is to provide a solution compatible with both interfaces.

In these new variables (more conveniently, their exponentials, (ξ, v)) perturbation theory is especially simple for just one interface. Together with (Γ) expressed perturbatively, we are able to fully integrate the perturbed p.d.e.'s for both interfaces, and reduce the problem to a purely algebraic one. We explicitly did so here for the first two orders of perturbation, and to first order wrote down all possible solutions. The most striking such solutions are just the usual infinite fluid pole dynamical ones, but decorated with exponentially small image terms, vanishing with $L \to \infty$. Since we already know that there are no such solutions, we wonder how they perturbatively appear so naturally and fluently. Before discussing this conundrum, we notice something central to the heart of the finite problem.

A finite body of viscous fluid has the natural constituentive relationship of impedance, the ratio of the net pressure across the fluid to the flux of its transport. Although impedance is infinite in the usual infinite fluid problem, it is less clear that its time derivative should also exactly vanish, as it does in the theoretical literature, but ostensibly in disagreement with experiment. This is a signature that the infinite boundary geometry is unphysical, even construed as the limit $L \to \infty$. Should a deformation of the fluid's flow produce a change in impedance, that is to say a change in the conserved flux through the entire fluid when pressure across it is maintained fixed, then such a deformation requests power from the pump that maintains this pressure difference. That is, such a deformation is *driven* by the pump, is controllable by the pump, and is certainly not spontaneously impressible. Such a circumstance is hardly surprising since a deformation is not just on the driven interface, but throughout the incompressible fluid that must everywhere adjust to accommodate to a distortion of its boundary. This is very different in the infinite fluid geometry, where the pole at infinity simply picks up whatever need be, since for the Riemann disc, the disposition of the one interface uniquely determines everything.

Performing the perturbative analysis of the finite fluid, we are forced to immediately decide if the perturbation is to modify flux, and so be externally driven. Should this not be allowed, we discover that the unperturbed finger has only purely stable modes, although not enough to allow for the arbitrary independent perturbations of both interfaces. The residual modes, all instabilities, are all driven by the external pump, and so, in no sense free fluctuations. These same modes, as $L \to \infty$, are precisely all the modes of the infinite fluid problem, although in that configuration they are all free. This now cements the observation that $L \to \infty$ is singular, and the infinite fluid geometry a physically wrong limit. There is no reason to doubt that these exponentially growing modes are real—they are, however, controllable. Indeed, a record of the flux through, or the power delivered by the pump over the course of time uniquely determines the disposition of this unstable flow throughout the entire body of fluid, and throughout this interval of time.

Granting these modes and an experimental control that allows them to grow at most algebraically in time, we now ask about the slow modes produced by summing all these fast modes. We especially care about summing them into logarithms, which then modifies the linear change of impedance in time, which then modifies the efflux transport from $L-\varphi$ to $L-(1/\lambda-1)\,\varphi$, hence modifies λ from $1/2$, and so, with no surface tension on the efflux produces a flow that violates $1-2\lambda=\sigma/v$. Here, resorting to (Γ), we discover that these logarithmic modes are in fact purely formal, where with ϵ the strength of perturbation, they fail to be solutions for any finite value of ϵ, no matter how small, thus resolving the conundrum that they don't exist in any correct non-perturbative analysis. With this, we establish that the solution of $\lambda=1/2$ is definite and isolated in the neighborhood of the pure unperturbed finger, and that we had not posed too brittle a question in asking for exact translation. Evidently, one would far prefer a purely non-perturbative treatment for a problem with a rather subtle, if not dangerous resolution.

Since impedance and the spatial shape of the interface are uniquely related for the unstable modes near the purely translating finger, the control of impedance can be replaced by material on the interface itself that controls shape variations that are rapid in space, such as a surface tension on the interface. Then the proto-selected pattern of the "pure" problem is stabilized, and becomes the observable pattern. That is, surface tension never determined a pattern. Rather it elected what had already been determined in the small by symmetry.

APPENDIX A. SELECTION WITHIN POLE DYNAMICS

The φ-translation invariant solutions of (3.37), which we shall here refer to as $\hat{f}_n$ with n as in that equation, belong to a certain well-studied class of solutions (Class (27) of ref. 1) We used no properties of that Class, but rather deduced (3.37) from dynamical symmetry arguments. This Class has a most important property: its members are exact solutions of (2.14′). These particular dynamical solutions are the so-called "pole dynamics." We shall, to render this paper more self-contained, re-derive these results here, as they are so trivial under reflection-symmetry, but then easily extend them to more complex variants. Most importantly, we shall see that even within this significantly enlarged class of pole dynamics, the *only* solutions for both interfaces within the class are precisely the $\hat{f}$'s of (3.37). That is, there exist no perturbations to $\hat{f}$ whatsoever in this general class, which is to say that had the dynamics been exhausted by pole dynamics, then $\hat{f}$ is rigidly isolated as *the* solution, and hence sharp pattern selection.

We start with the delineation of pole dynamics (Class (27)):

$$f = \beta(\varphi) + \zeta + \Sigma\alpha_k \ln(1 - e^{\zeta_k - \zeta}); \qquad \operatorname{Re} \zeta_k(\varphi) < 0; \qquad \alpha_k = \text{const} \tag{A.1}$$

where f must satisfy (2.14′) and be reflection-symmetric (β real; (α_k, ζ_k), $(\bar{\alpha}_k, \bar{\zeta}_k)$ both present). Paying attention to (2.14′), write

$$f' = 1 + \Sigma \frac{\alpha_k}{e^{\zeta - \zeta_k} - 1}; \qquad f_\varphi = \beta' - \Sigma \frac{\alpha_k \zeta'_k}{e^{\zeta - \zeta_k} - 1}. \tag{A.2}$$

Consider $\zeta \to \zeta_k$. Then $1/f'(\zeta) \to 0$, $f_\varphi / f'(\zeta) \to -\zeta'_k(\varphi)$. Dividing (2.14′) by $f'(\zeta)\, f'(-\zeta)$ and taking $\zeta \to \zeta_k$,

$$0 = -\zeta'_k + f_\varphi(-\zeta_k)/f'(-\zeta_k);$$

$$0 = f_\varphi(-\zeta_k, \varphi) - \zeta'_k(\varphi)\, f'(-\zeta_k, \varphi) = \frac{d}{d\varphi} f(-\zeta_k, \varphi) \tag{A.3}$$

$$\Rightarrow f(-\zeta_k, \varphi) = \bar{z}_k = \text{const.}$$

These points, $z_k = f(-\bar{\zeta}_k, \varphi)$ are "stagnation" points, in the sense that as $\varphi \to \infty$ all Re $\zeta_k \to 0^-$, so that z_k is very near to a point on the interface $\zeta = is$, and the interface may never pass through this point. More correctly, the interface must asymptotically come to rest at

$$z_k^* = z_k - \alpha_k \ln 2 \tag{A.4}$$

since as Re $\zeta_k \to 0$, $-\bar{\zeta}_k \to \zeta_k$, and f is singular at ζ_k. Setting $\zeta = i\eta_k + it\xi_k$ with $\zeta_k = \xi_k + i\eta_k$, writing down $f(\zeta)$, $f(-\bar{\zeta}_k)$, and subtracting yields

$$f(i\eta_k + it\xi_k) \sim z_k - \alpha_k \ln 2 + \alpha_k \ln(1 - it). \tag{A.5}$$

For a flow within an initially almost flat interface that grows wrinkled, all the z_k have real parts to the right of the interface, and so, by (A.4) must have Re $\alpha_k > 0$, since the interface can never have passed through z_k itself.

(A.3) determines all the $\zeta_k(\varphi)$ in terms of β. f has derivatives analytic at Re $\zeta \to \pm\infty$: By (A.2),

$$\begin{aligned} f'(+\infty) &= 1, \qquad & f'(-\infty) &= 1 - \Sigma\alpha, \\ f_\varphi(+\infty) &= \beta', \qquad & f_\varphi(-\infty) &= \beta' + \Sigma\alpha_k\zeta'_k, \end{aligned} \tag{A.6}$$

and so by (2.14′),

$$(1 - \Sigma\alpha)\,\beta + (\beta + \Sigma\alpha_k\zeta_k) = 2\varphi + \text{const}, \tag{A.7}$$

thus fully determining f.

As $\varphi \to \infty$, $\zeta'_k \to 0$, as Re $\zeta_k \to 0^-$, and by f's imaging, Im $\zeta_k \to 0$, π only (see Paper I). By (A.2) outside arbitrarily small disks about the ζ_k, $f_\varphi \sim \beta'$, and by (2.14′),

$$\frac{2}{\beta'} \sim f'(\zeta) + f'(-\zeta)$$

with $f'_\varphi \sim 0$, so $f'(\zeta, \varphi) \sim f'(\zeta)$. Hence

$$\beta' \equiv \frac{1}{\lambda} = \text{const}, \tag{A.8}$$

and $f'(\zeta) + f'(-\zeta) \sim 2\lambda$, $f(\zeta) - f(-\zeta) \sim 2\lambda\zeta$ and so on $\zeta = is$,

$$y(s) = \text{Im}\, f(is) \sim \lambda s \qquad s \in (\epsilon, \pi - \epsilon) \tag{A.9}$$

which is to say it is a finger of width λ of the channel. By (A.7) and (A.8)

$$\lambda - \tfrac{1}{2} = \tfrac{1}{2}(1 - \Sigma\alpha) = \tfrac{1}{2} f'(-\infty). \tag{A.10}$$

Now, the $\hat{f}_n$ of (3.37) are explicitly of form (A.1) with $f'(-\infty)=0$, and so $\lambda=1/2$. We ultimately care about $\hat{f}_1$, a single 1/2 finger in a 2π channel. It is now natural to ask if there are other Class (27) solutions for both interfaces, since then we could analytically solve the problem of such perturbations to $\hat{f}$. It is easy to see that there are none.

First, substitute (2.15) into (2.14″) to obtain

$$\begin{aligned} 2-f'(-\zeta+\xi_g)\, f_\varphi(\zeta+\xi_g)-f'(\zeta+\xi_g)\, f_\varphi(-\zeta+\xi_g) \\ =2\xi_g' f'(\zeta+\xi_g)\, f'(-\zeta+\xi_g) \end{aligned} \tag{A.11}$$

with ξ_g finite, taking the limit $\operatorname{Re}\zeta\to+\infty$, where Class (27) is analytic, the left hand side vanishes, as it is the limit of simultaneous (2.14′). Hence the only f's that are allowed satisfy

$$\xi_g'(\varphi)\, f'(+\infty)\, f'(-\infty)=0. \tag{A.12}$$

But $f'(+\infty)=1$, and for any non-flat solution $\xi_g'\neq 0$, and so $f'(-\infty)=0$, $\lambda=1/2$, and so,

$$\Sigma\alpha=1 \tag{A.13}$$

is a consistency condition for f to obey both equations of motion. Thus, $\lambda=1/2$ is selected.

There is an important simple fact related to why $\xi_g'\neq 0$. There is a unique conformal (i.e., invertible) map h from the fluid at time φ to a rectangle with sides ξ_g and 2π. The aspect ratio, $\xi_g/2\pi$, of this rectangle is termed the "module" of the region mapped, and is uniquely determined by that region. Moreover, it enjoys an exact estimate:

$$2\pi\xi_g(\varphi)\leqslant A(\varphi)\equiv A_0$$

with A the area of the fluid, the constant in time A_0. Unless $h'\equiv 1$, the inequality is strict. Thus, with $A_0\equiv 2\pi L_0$, we have

$$\xi_g(\varphi)\leqslant L_0.$$

That is, for other than the flat flow $\xi_g(\varphi)$ is strictly smaller than the resting length of the fluid. In particular, for a solution which was flat in the far past, $\xi_g'\neq 0$, since it had the value L_0 in the far past, but certainly below it at finite times.

With f of (A.1), g of (2.15) is

$$g=(\beta+\xi_g)+\zeta+\Sigma\alpha_k\ln(1-e^{\zeta_k-\xi_g-\zeta}). \tag{A.14}$$

Between f and g, the $\zeta \to \infty$ equation, (A.7) is identical with $\Sigma\alpha = 1$. So, there is one such equation for β. However together with (A.3), we now also have

$$g(\xi_g - \bar{\zeta}_k, \varphi) = z_k^+. \tag{A.15}$$

With a real ζ_k, by f's reflection symmetry such a (A.3), (A.15) is a real equation. For each complex ζ_k, each is complex, and hence a pair of real equations. So, with m complex ζ_k's and n real ones, we have

$$\#\text{ real eq's} = 4m + 2n + 1 \tag{A.16}$$

and

$$\#\text{ real vbl's} = 2m + n + 2 \qquad (\zeta_k\text{'s} + \xi_g + \beta) \tag{A.17}$$

and so the system is overdetermined by

$$\text{excess} = 2m + n - 1. \tag{A.18}$$

So, save for the case of just one $\zeta_k \equiv \hat{\zeta}$ which is real, and even with just one complex $\hat{\zeta}$, the system is overdetermined. The one real case is just $\hat{f}_1$ of (3.38).

Let us now show that the only solutions that become flat as $\varphi \to -\infty$ are just the $\hat{f}_n$, with the overdetermination in fact inconsistent in all other cases.

Writing down (A.3) for a ζ_k and the difference of (A.15) and (A.3), we have

$$\bar{z}_k = \beta - \zeta_k + \Sigma\alpha_\ell \ln(1 - e^{\zeta_l + \zeta_k}) \tag{A.19}$$

$$\bar{z}_k^+ - \bar{z}_k \equiv \Delta_k = 2\xi_g + \Sigma\alpha_\ell \ln\left(\frac{1 - e^{-2\xi_g + \zeta_k + \zeta_\ell}}{1 - e^{\zeta_k + \zeta_\ell}}\right). \tag{A.20}$$

For all Re $\zeta_k \to -\infty$, $(\varphi \to -\infty)$, $\xi_g \to \xi_0$ (finite), we immediately have by (A.20)

$$\Delta_k \equiv 2\xi_0. \tag{A.21}$$

Expanding (A.20), since all Re $\zeta_k < 0$ and $\xi_g > 0$ and so the series are absolutely convergent, produces

$$\Sigma\alpha_\ell \sum_{r=1}^{\infty} \frac{1}{r} (1 - e^{-2r\xi_g})\, e^{r(\zeta_k + \zeta_\ell)}$$

which can be resummed as

$$\sum_{r=1}^{\infty}\left[\frac{1}{r}(1-e^{2r\xi_g})(\Sigma\alpha_\ell e^{r\zeta_\ell})\right]e^{r\zeta_k}.$$

So,

$$\begin{aligned}2(\xi_0-\xi_g)&=\sum_{r=1}^{\infty}u_r(\varphi)\,e^{r\zeta_k};\\ u_r(\varphi)&\equiv\frac{1}{r}(1-e^{-2r\xi_g})\,\Sigma\alpha_\ell e^{r\zeta_\ell}.\end{aligned}\tag{A.22}$$

The first of (A.22) implies that

$$u_r(\varphi)\equiv 0\quad r<n;\qquad u_n\neq 0\quad\text{and}\quad e^{n\zeta_k}\equiv e^{n\zeta_0}\quad\text{for some}\quad n\geqslant 1.\tag{A.23}$$

Thus

$$\zeta_k=\zeta_0+2\pi i\,\frac{k}{n}\qquad k=0,\ldots,n-1.\tag{A.24}$$

For $u_{r<n}\equiv 0$, by the second of (A.22), $\sum_{\ell=0}^{n-1}\alpha_\ell e^{2\pi i\frac{r\ell}{n}}=0$, $r=1,\ldots,n-1$, and so by finite Fourier, with (A.13),

$$\alpha_k\equiv 1/n.$$

Finally, taking Re ζ sufficiently large and analytically continuing the result,

$$\begin{aligned}f&=\beta+\zeta+\frac{1}{n}\sum_{k=0}^{n-1}\ln(1-e^{\zeta_0-\zeta}e^{2\pi ik/n})\\ &=\beta+\zeta-\sum_{r=1}^{\infty}\frac{1}{r}\,e^{r(\zeta_0-\zeta)}\left(\frac{1}{n}\sum_{k=0}^{n-1}e^{2\pi ikr/n}=\sum_{\ell}\delta_{r,n\ell}\right)\\ &=\beta+\zeta-\frac{1}{n}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\,e^{n\ell(\zeta_0-\zeta)}\\ &=\beta+\zeta+\frac{1}{n}\ln(1-e^{n(\zeta_0-\zeta)}).\end{aligned}\tag{A.25}$$

By reflection-symmetry $e^{n\zeta_0}$ is real, or $e^{n\zeta_0} = \pm e^{n\hat{\zeta}}$ for $\hat{\zeta}$ real, and so

$$f = \beta + \hat{\zeta} + \frac{1}{n} \ln(e^{n(\zeta - \hat{\zeta})} \pm 1)$$

which then to satisfy (2.14′) is just $\hat{f}_n$ of (3.37). It must be remembered that (2.14′) is nonlinear: the only f's of Class (27) is precisely *one* of these $\hat{f}_n$'s *not* any linear combinations whatsoever. That is, there can be *no* Class (27) perturbations to $\hat{f}$, so that within this Class, $\hat{f}$ is perfectly stable of unique 1/2 width.

Now, Class (27) arose in discussions of the infinite channel: in the natural variable $e^{-\zeta} \equiv \omega$, $u = e^{-f}$ is analytic at $\omega = 0$, where it has a simple zero. e^{-f} is the unique (under reflection symmetry) Riemann map taking $u = 0$ to $\omega = 0$ of the physical region to the unit disk. With two free boundaries, the map is to the annulus $e^{-\xi_g} < |\omega| < 1$, and so is generally not analytic in $|\omega| < e^{-\zeta_g}$. The f's of (A.3) are a Taylor expansion about $\omega = 0$. For two free boundaries, it is generally to be a Taylor–Laurent expansion. Let us write this as

$$f = \beta + \zeta + \Sigma \alpha_k \ln(1 - e^{\zeta_k - \zeta}) - \Sigma \alpha_k^+ \ln(1 - e^{-\zeta_k^+ + \zeta})$$
$$\text{with} \quad \mathrm{Re}\, \zeta_k < 0, \quad \mathrm{Re}\, \zeta_k^+ > 0. \tag{A.26}$$

To fully consider these solutions, we extend the pole-dynamics of (A.3) to include the special circumstance that the ζ_k and ζ_k^+ might be matched with $\zeta_k^+ \equiv -\zeta_k$, which is the "closest" that (A.26) can come to a solution. (There are none.)

Writing the derivatives,

$$\begin{aligned} f' &= 1 + \Sigma \frac{\alpha_k}{e^{\zeta - \zeta_k} - 1} + \Sigma \frac{\alpha_k^+}{e^{\zeta_k^+ - \zeta} - 1}; \\ f_\varphi &= \beta' - \Sigma \frac{\alpha_k \zeta_k{}'}{e^{\zeta - \zeta_k} - 1} - \Sigma \frac{\alpha_k^+ \zeta_k^{+\prime}}{e^{\zeta_k^+ - \zeta} - 1}. \end{aligned} \tag{A.27}$$

First

$$\begin{aligned} f'(+\infty) &= 1 - \Sigma\alpha^+, & f'(-\infty) &= 1 - \Sigma\alpha, \\ f_\varphi(+\infty) &= \beta' + \Sigma\alpha_k^+ \zeta_k^{+\prime}, & f_\varphi(-\infty) &= \beta' + \Sigma\alpha_k \zeta_k', \end{aligned} \tag{A.28}$$

so that (2.14′) yields

$$(1 - \Sigma\alpha)(\beta + \Sigma\alpha_k^+ \zeta_k^+) + (1 - \Sigma\alpha^+)(\beta + \Sigma\alpha_k \zeta_k) = 2\varphi + \text{const.} \tag{A.29}$$

It is next immediately clear that (A.3) is unmodified for both ζ_k *and* ζ_k^+ *if* they are unpaired:

$$f(-\zeta_k, \varphi) = \bar{z}_k, \qquad f(\zeta_k^+, \varphi) = \bar{z}_k^+, \qquad \zeta_k + \zeta_k^+ \neq 0. \tag{A.30}$$

Should $\zeta_k + \zeta_k^+ = 0$ (arrange the indices so that mating ζ's have the same index), then the derivation of (A.3) fails. Instead we compute with (A.27) $f'(\pm(\zeta_k + \epsilon))$ and $f_\varphi(\pm(\zeta_k + \epsilon))$, and proceeding with a little care determine that

$$\bar{z}_k = \alpha_k f_R(-\zeta_k, \varphi) + \alpha_k^+ f_R(\zeta_k, \varphi) \tag{A.31}$$

where f_R means drop the term that diverges at the specified argument of f *precisely* of form (A.26). Most importantly, there is just *one* equation (A.31) for the matched pair $\pm\zeta_k$ which lowers the overdetermination count of (A.18).

The problem is that it doesn't lower the count enough, since if a pair matches in f it can't in g and visa versa: The singularities of g, by (A.26) are at $\zeta_k - \xi_g$ and $\zeta_k^+ - \xi_g$, so that if ζ_k and ζ_k^+ are paired, and hence with mean 0, in g they have mean $-\xi_g$ and are no longer matched, and so each produces its own equation (A.15). This, together with the notion of the "method of images," determines the best possible way to place f's singularities. Consider a real

$$\zeta_0 \equiv \hat{\zeta} < 0. \tag{A.32}$$

With $-\hat{\zeta} > 0$ and approaching 0 as $\varphi \to +\infty$, it is almost impossible for $-\hat{\zeta}$ not to be in physical fluid, and so take it as unmatched. Next consider more ζ_k:

$$\zeta_k = \hat{\zeta} - 2k\xi_g \qquad k = 0,..., n. \tag{A.33}$$

But then

$$\zeta_k^+ \equiv -\zeta_k = -\hat{\zeta} + 2k\xi_g \qquad k = 1,..., n \tag{A.34}$$

are all matched, greater than $2\xi_g$, so certainly outside physical fluid. These are f's singularities. For g, we have

$$\zeta_k^g \equiv \zeta_{k-1} - \xi_g = \hat{\zeta} + \xi_g - 2k\xi_g \qquad k = 1,..., n+1 \tag{A.35}$$

and

$$\zeta_k^{+g} = \zeta_k^+ - \xi_g = -\hat{\zeta} - \xi_g + 2k\xi_g = -\zeta_k^g \qquad k = 1,..., n. \tag{A.36}$$

Again all the ζ_k^g are matched, save for ζ_{n+1}^g. Thus both f and g have n matched pairs and one unmatched singularity. (This is the best that can be arranged within the ξ_g annular domain.) Counting, this is $2n+3$ real equations for $2n+1$ singularities, ξ_g and β. This has a right sense to it, save for the difficulty that the $2n+1$ singularities are just combinations of $\hat{\zeta}$ and ξ_g, so that there are precisely always 3 variables, and so, overdetermined save for $n=0$, which is simply $\hat{f}_1$. It should be reasonably clear that f of (A.26) affords no exact solutions other than the $\hat{f}_n$ of φ-translation invariance. That is, there is *no* pole dynamics at all. Rather, that class, natural to the dynamics, truly arises as a guessed-at extension of the Saffman–Taylor stationary solution, but exists only for $1/2$ fingers in consequence of displacement invariance.

That is, the $1/2$ finger has unlimited stability and selection within the scope of any conceivable pole dynamics. Insofar as the finite-time singularities in the literature are all of pole dynamics, $\hat{f}_1$ is evidently totally free from them.

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